INTRODUCTION TO
REAL ANALYSIS

Michael J. Schramm

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MICHAEL J. SCHRAMM
Le Moyne College

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Preface

This text is an introductory course in real analysis, intended for students who have completed a calculus sequence. It is also designed to serve as preparation for advanced mathematics courses of many sorts. Though there are occasional references to it in exercises, linear algebra is not specifically a prerequisite for this text. Nevertheless, the changing role of linear algebra in the undergraduate curriculum is one of the main reasons this book comes to be the way it is. In the past, a first course in linear algebra was generally considered to be the place where one "learned to do proofs." The mathematics curriculum has gradually changed, though, and proofs as such are no longer the main focus of the typical linear algebra course. As a result, a student's first extensive experience with the logical and organizational skills necessary for the successful construction of proofs is often delayed until they find themselves in courses in which success is predicated on possession of those very skills.

Textbooks have been slow to adapt to these changes. This book provides a pathway from the calculus course to real analysis (and beyond) in which the discussion of the construction of proofs is a continuing and central theme. Throughout the text (but most especially in Part 2), proofs are not simply presented in final form. Rather they are shown as works-in-progress; they are *built*, and their construction is discussed along with their final content. While learning real analysis, the student will, it is hoped, also learn something about the workings of the mind of a mathematician—invaluable information if one is to become a researcher in or a teacher of mathematics, or both, but information that is often overwhelmed by the demands of the subject at hand.

The text is organized into four parts. The material of Part 1 is the common foundation of most upper-level mathematics courses. The book begins with an introduction to basic logical structures and techniques of proof. The ideas introduced here, especially the crucial "forward-backward method" of constructing proofs, are all emphasized and used explicitly throughout the text. The rest of Part 1 includes discussions of the concept of cardinality, the algebraic and order structures of the real and rational number systems, and the natural numbers in their dual role as the basis for induction and as special elements in ordered fields. The discussion of the real and rational number systems sets the stage for

Part 2 of the book. In Part 1 it is found that these systems have much in common, while Part 2 is devoted to discovering how they are different.

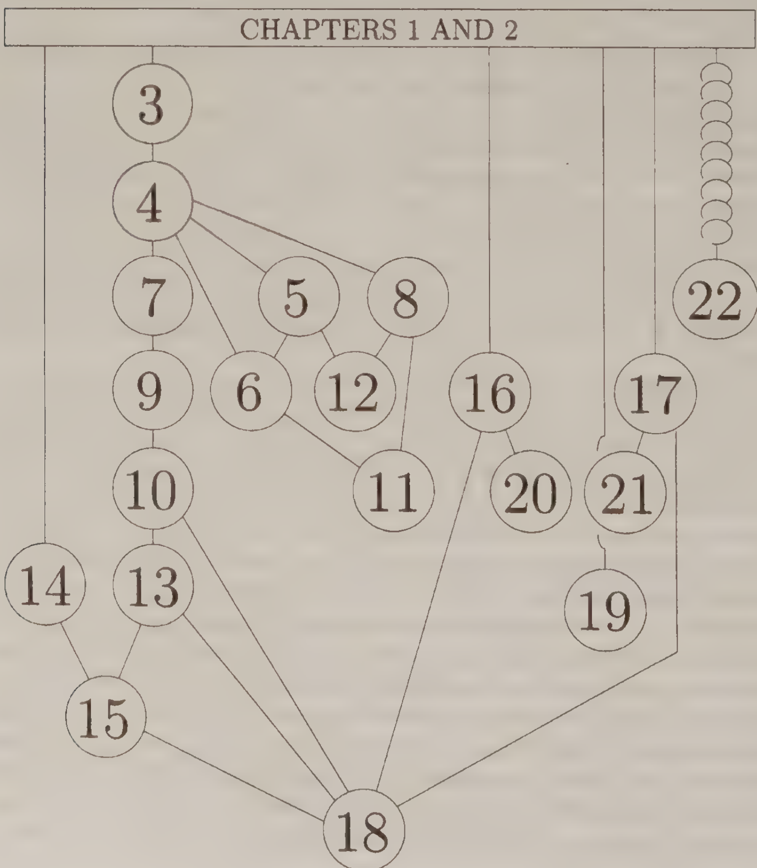
Part 2 of the text is an in-depth examination of the completeness of the real number system in its various guises and a discussion of the topological structure of the real number system. It is here also that the discussion of the construction of proofs becomes most focused. The student is regularly reminded of the relationship between the topic at hand and the larger story by a graphic device called the "Big Picture." The Big Picture, which describes interrelationships among the various manifestations of the completeness of the real number system, gives a structure and unity to the subject not found in most mathematics courses. The organization of Part 2 is such that after its first four chapters are discussed, the rest may be treated in any order and at the discretion of the instructor. The structure provided by the Big Picture allows the option of showing that the important properties of the real number line do not exist in isolation; they are in fact equivalent.

Part 3 of the text is a review and extension of calculus in light of the student's new understanding of the real number system. With the knowledge and experience gained in Part 2, the student can appreciate the structures and results of calculus to a depth not possible in a first-year course, and is prepared to deal with proofs presented in a more direct and traditional manner. Part 4 is a selection of topics in real function theory, investigated as natural outgrowths of questions readily understood by the student. The instructor has great latitude to choose topics in the second half of the text, and coverage of this material can range from an intensive reconstruction of calculus to a discussion approaching current research topics.

The exercises in the text are many and varied. They range from the fairly routine—checking steps omitted from examples and observing mechanical results at work—to completion of partial proofs from the text to extended projects, some of which border on current research. There are questions in which the student is asked simply to discuss a statement, or to explain to their own satisfaction why something is true or false, free from the restrictions of a formal proof. There are questions in which the student is asked to find flaws in incorrect (though possibly convincing) "proofs," and questions in which the student is asked to reconsider their own proofs in light of new information. Most importantly, the exercises are an integral part of the text. Regular and meaningful cross-referencing reminds the student of the unity of the subject and highlights their own active role in its development. Furthermore, the exercises themselves constitute part of the ongoing study of the workings of the mathematical mind, as the student is often led from one topic to another in a way that suggests, it is hoped, that no matter how many answers one finds, there

are always more questions to be asked.

The diagram below may be interpreted like this: The overall flow of the subject is from top to bottom. The material of Chapter 1 supports everything else, and the ideas in Chapter 2 pop up just about everywhere. Strong dependence between chapters is indicated by a line, though ideas from a chapter represented by a higher ball might be needed in one below it (spatially or structurally). For instance, it is necessary to understand the material of Chapter 4 to make sense of Chapter 5, but Chapter 4 is less directly needed in Chapter 12 and even less so in Chapter 20. Aside from experience, there are no prerequisites for Chapters 14 and 19, and Chapter 22 may be taken up any time after Chapter 5 (hence its variable position). Part 2 of the book (Chapters 5 through 12) has an internal structure of its own, which is described in the introduction to that section.



Gratitude, like proofs, should sometimes follow the forward-backward method. I am eternally grateful to Professor Ronald Shonkwiler of the Georgia Institute of Technology for starting me on my way to becoming a mathematician, and to Professor Daniel Waterman and the late David Williams of Syracuse University for doing their best to help me finish the journey. In the present, I am indebted to Jim Spencer of the University of South Carolina at Spartanburg and Robert E. Zink of Purdue University, and others whose identities I will never know, for their careful and thoughtful reviews of the manuscript of this book. Their suggestions have much improved the end result. For the future, I wish to thank the students who have been so tolerant during the development of this text. They have, with the unerring radar available only to those to whom a subject is new, corrected shocking slews of errors, and have, for the most part, helped me overcome my tendency to write questions in which it is necessary to do part (b) before part (a). They have also, much as was hoped, asked lots of questions, and many of the exercises in the text were proposed by students in the course. It can be said with equal validity in regard to all three of these groups of people, that most of what is good here is theirs, while the remaining errors and oversights are mine alone.

Part One

Preliminaries

We begin our work with a discussion of the construction of proofs. Writing proofs, of course, is the heart of mathematics. If there were always direct, mechanical processes for writing proofs, though, mathematics would not be nearly so fascinating. We will find that proofs need not be as mysterious as they might seem and that we can smooth the way considerably by making use of some basic organizational schemes.

The rest of the first half of the book is best understood by thinking of what we will call the **Big Question**: How are the real numbers different from the Rational Numbers? We will tackle this in two stages. In Part 1 of the book we will see some of the ways these two number systems are similar. It's best to know the ways things are alike before asking how they are different. In Part 2 of the book, we examine the differences between them and answer the Big Question. Our efforts will be richly rewarded. We will find that the property that distinguishes the real number system from the rational number system is precisely what makes calculus work.

Through all of this we will never say what the real numbers actually are! That we can consider working this way is one of the remarkable features of mathematics. We can study how the real numbers *work*, blissfully unconcerned with what they *are*. We can solve the crime, so to speak, without ever knowing the suspects. We will finally meet the real numbers at the very end of the book. Agatha Christie would be proud.[1] If we don't even know what they are, how can we hope to say that the real numbers are different from the rational numbers? Like this: If an object X possesses some mathematical property that the object Y does not, we can say with confidence that X is different from Y. If X is definitely red, Y is definitely blue, and (this is most important) a red thing can't also be blue, then X and Y must be different. This sort of argument underlies much of what happens in this book. Be sure to watch for it.

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[1] Agatha Christie annoyed faithful readers of her wonderful mysteries for decades by revealing essential clues only in the final scene ("I suppose you're wondering why I've called you all here ...").

Chapter 1
Building Proofs

1.1 A QUEST FOR CERTAINTY

The study of mathematics is the quest for a sort of certainty that can be attained in no other endeavor. In mathematics we can "prove" things. But what does this mean? Less than we might hope. Bertrand Russell, one of the foremost British philosophers of recent times, called mathematics "the subject in which we never know what we are talking about, nor whether what we are saying is true." If this is the case, how can we hope to prove anything? We can't! What we *can* do, however, is show with absolute certainty that each of a chain of statements is "as true as those before it." If we believe the statements at the beginning of the chain, and that the chain is properly assembled, we *must* believe the statements at the end.[1]

Of course, we have to begin somewhere, and it is evident that statements at the beginning of such a chain *can't* be proved. Statements that we agree to accept without proof are called **axioms**. We may discuss whether an axiom is appropriate (that is, whether it describes life as we perceive it) and we might at some point want to spend time discussing which axioms we ought to believe and which we should reject. But once this issue has been settled (and for the purposes of this course we consider it to be so) we agree *not* to discuss whether an axiom is true or false. Though it certainly can be an activity of great value, it is not our goal to scrutinize a collection of axioms here. We are studying the "top floors" of a subject, not its "foundations." Besides, the chain of reasoning leading from the most basic axioms to this text is unimaginably long. Bertrand Russell and Alfred North Whitehead took it upon themselves to build such a chain in their monumental *Principia Mathematica*. After several *hundred* pages, they were able to prove from "first principles"

[1] This is the price mathematicians pay for the power to make proofs. In mathematics we find things that we are *compelled* to believe (some of which we might prefer not to). Practitioners of other fields have more flexibility to pick and choose what they will accept.

that $1+1=2$. Fortunately, this is not how we will be spending our time. Foundational questions are as much philosophical as mathematical, and as mathematics they fall under headings other than analysis.

EXERCISES 1.1

1. What did Russell mean when he said that mathematics is "the subject in which we never know what we are talking about, nor whether what we are saying is true"? we can't truly prove anything about mathematics
2. Discuss the differences in meaning of a statement of the form "It is true" when the assertion is made by a mathematician, a physicist, a biologist, a sociologist, a politician, or a used-car salesperson. different principles

1.2 PROOFS AS CHAINS

It is instructive in many ways to view a proof as a chain of reasoning. To build a chain, we need a supply of links and a way to connect one link to another. In geometry class, we sometimes made two-column proofs with "Steps" on one side of the paper and "Reasons" on the other. A step might have been "Angle a is congruent to angle b," with the reason "Alternate interior angles." Steps are the links in the chain; reasons are the connections between them. We can safely use a real chain only if each of its links and the connections between them are sound. In mathematics, a sequence of statements, each of which is properly formed and correctly justified by those before it, is called a **proof**.

We can construct a chain from either end, or from both ends at once. We can even assemble links bound for the middle into sections and then connect the sections. In the same way, we can work a proof from the beginning or the end (or even from the middle). A proof is almost never thought of straight through from beginning to end. In textbooks, though, proofs are usually *written down* from beginning to end, causing much unnecessary confusion. Here we will give a brief outline of the basic logical structures we will encounter in our work. We observe right off the bat that even the simplest of ideas sometimes warrant discussion.

1.3 STATEMENTS

We have already used this important word, even though we may not be entirely sure what it means. Since statements are the steps in our proofs—the links in our chains—we should examine the meaning of the

word carefully. Unfortunately, it is a bit difficult to capture, and the results might be a bit unsatisfying: **A statement** is a grammatically meaningful sentence to which one or the other (but not both) of the words "true" or "false" can be attached. The appropriate one of these is called the **truth value** of the statement. We see that "$1 + 1 = 2$" is a statement, since it may be labeled "true," and that "$1 + 1 = 3$" is a statement because it is "false." A collection of words to which no truth value can be attached is simply *not a statement.* For instance, consider the phrase "This sentence is false." If we believe this to be true, then the assertion it seems to make is true, and consequently it is false. On the other hand, if we believe it to be false, the assertion it seems to make is false, and so the sentence must be true! However we view it, we are led to conclude that the phrase is both true and false, which cannot be. We resolve this paradox by saying that "This sentence is false" is *not a statement.* (We have hedged our bets by saying that the sentence *seems* to make an assertion. Since it is not a statement, it can't make any assertion at all.)

The discussion of which collections of words are statements and which are not is another subject, and we won't go into it here. It will be enough for our purposes to note that a statement must *assert* something. This means, among other things, that *a statement must contain a verb* (in mathematics the verb is often $=$). Here is a very simple (but remarkably useful) preliminary test to check whether a collection of words is a statement:

IF IT DOESN'T MAKE SENSE AS LANGUAGE,
IT DOESN'T MAKE SENSE AS MATHEMATICS.

The very best way to check this is to *read* what you write, preferably out loud. If your writing doesn't *sound* meaningful, it isn't. Many of what seem to be errors in understanding are actually only errors in grammar. This principle is most often violated in the writing of sentences that make no assertion. Sentences with no verbs!

EXERCISES 1.3

1. Decide whether the following are statements:

   (a) $2 + 4 = 7$. ✓

   (b) $\sin^2 x + \cos^2 x = 1$. X

   (c) This sentence no verb. X

(d) The sequence of digits 0123456789 appears somewhere in the decimal expansion of π.([2]) ✓

1.4 CONNECTIVES

There is a limit to the depth of ideas that can be expressed in statements like "$1 + 1 = 2$" and "A pencil is a writing utensil" (these are called **simple statements**). More interesting are **compound statements** like "Either $1 + 1 = 2$ or a pencil is a useful tool in neurosurgery" and the more mundane "A subset of the real line is closed if and only if its complement is open."

The tools with which we make compound statements out of simpler ones are called **connectives**, of which we need to consider only four. The simplest connective is **negation**: If "A" is a statement, so also is "not A," which is sometimes denoted $\neg A$ (we will use the word instead of the symbol). Since A is a statement, it has a truth value. The statement "not A" is assigned the truth value not given to A. So "$1 + 1 = 2$" is a (true) statement whose (false) negation is "not $(1 + 1 = 2)$." Here we could also say "$1 + 1 \neq 2$," but it often takes some effort to phrase the negation of a statement in ordinary language. (Since it doesn't really "connect" anything, negation is often referred to as a **modifier** rather than a connective.)

Our first genuine connective is **conjunction**. If A and B are statements, their conjunction is the statement "A **and** B" (or "$A \wedge B$"). We may describe the truth values of "A and B" with a **truth table** like the one below:

A	B	A and B
T	T	T
T	F	F
F	T	F
F	F	F

All combinations of truth values of the two statements A and B appear in the first two columns of this table. The last column tells us that "A and B" is true only when A and B are both true. This agrees with our usual understanding of the word. (But the meaning of "and" is being *defined* in this table. It need not coincide with ordinary usage, though it

[2] Whether the last of these is a statement is actually open to debate. Most mathematicians would agree that it certainly is either true or false (all the while admitting they don't know which), but to a group known as "intuitionists" it is *neither*, despite its unambiguous form. See footnote 7 in this chapter.

is all the better if it does.) If we assert that "it is warm *and* sunny," we expect it both to be warm and to be sunny. By specifying truth values of the statement "A and B," we are also asserting that "A and B" is a statement as long as A and B are statements.

The next connective is **disjunction**. The disjunction of A and B is written "**A or B**" (or "$A \vee B$"), and is given by:

A	B	A or B
T	T	T
T	F	T
F	T	T
F	F	F

The mathematical "or" is not quite the same as the grammatical "or." In ordinary usage, "or" can mean "one or the other or both" and it can mean "one or the other *but not* both." The former is called **inclusive disjunction** and the latter **exclusive disjunction**. In mathematics "or" *always* refers to inclusive disjunction. Exclusive disjunction is not uncommon in everyday language ("You can eat your lima beans or you can skip dessert"), but we encounter it so seldom in mathematics that we don't have a separate term for it.

The most important connective is **implication**. We use implication to join the links in the "chains" that constitute our proofs. We write "$A \Rightarrow B$" and say "A **implies** B." Here A is called the **hypothesis** and B is the **conclusion**.[3] Implication is defined by this table:

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

This deserves more discussion. We may think of an implication as a rule that is true if it is being obeyed and false if it is being broken:

IF IT IS RAINING, THEN YOU MUST CARRY AN UMBRELLA
(RAIN $\Rightarrow$ UMBRELLA)

[3] There are several other ways of saying this: "if A then B"; "B follows from A"; "B if A"; "A only if B"; "A is sufficient for B"; "B is necessary for A." Note the individual meanings of the words "if" and "only if" and of "necessary" and "sufficient."

Suppose it's raining and you're carrying your umbrella. You are not breaking the rule, and all is well (because "true $\Rightarrow$ true" is true). If it's raining and you don't have your umbrella, you are breaking the rule ("true $\Rightarrow$ false" is false). If it's not raining, you are obeying the rule *whether you have your umbrella or not.* The rule is not being broken by "no rain and umbrella" (false $\Rightarrow$ true) or by "no rain and no umbrella" (false $\Rightarrow$ false). Mathematicians agree to consider "false $\Rightarrow$ true" and "false $\Rightarrow$ false" to be true, but they do so grudgingly. Such implications are said to be *vacuously* true.[4]

Analyzing compound statements in this way is a very small part of **symbolic logic**. We use symbolic logic to help us understand complex statements in terms of their (simpler) component parts. Sometimes it is possible to deduce the truth value of a compound statement from its *form* rather than its *content*. For example, "A or not A" is true regardless of the content or truth value of the statement A, while "A and not A" is always false (be sure you see why this is so).

These examples are too simple to illustrate the value of symbolic logic. We will examine some that are more significant. For instance, it might be useful to have an alternative means of expressing the relationship "$A \Rightarrow B$." Is there another statement containing the letters A and B that is false only when A is true and B is false? The "Umbrella Rule" would be false only if it were raining but there were no umbrellas in sight. This would correspond to "A and not B." Perhaps "not (A and not B)" is the same as "$A \Rightarrow B$." We can check this with a truth table:

A	B	not B	A and not B	not(A and not B)	$A \Rightarrow B$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	F	T	T
F	F	T	F	T	T

The two rightmost columns of this table tell us that the statements "$A \Rightarrow B$" and "not (A and not B)" have the same truth value for any combination of truth values of A and B. We may show "not (A and not B)" instead of "$A \Rightarrow B$" if the former is more convenient. If two statements X and Y always have the same truth value, we say that they are **logically equivalent** and write $X \Leftrightarrow Y$ or $X \equiv Y$. Were we to include

[4] Russell once gave a speech in which he asserted "From false premises I can reach any conclusion." A voice in the audience said: "Assume that $1 = 2$ and prove that you are pope." (Russell's opinions on religion made this a good joke.) Now the statement "If $1 = 2$ then B. R. is pope" is true without further explanation, but Russell rose to the challenge: "The pope and I are two, therefore we are one."

a column in the previous table for "$(A \Rightarrow B) \Leftrightarrow$ not (A and not B)," all its entries would be T. An expression that is always true is called a **tautology**. One that is always false is a **contradiction**. Symbolic logic, in the rudimentary form in which we use it, is a search for tautologies and contradictions. "$A \Leftrightarrow B$" is also read "A if and only if B" or "A is necessary and sufficient for B." You will show in Exercise 1.4.2 that "$A \Leftrightarrow B$" is equivalent to "$(A \Rightarrow B)$ and $(B \Rightarrow A)$," thus avoiding a possible conflict in meaning.[5]

EXERCISES 1.4

1. Make up more examples to illustrate the inclusive and exclusive "or."

2. (a) Show that "A or not A" is a tautology and "A and not A" is a contradiction.

   (b) Show that "$(A \Rightarrow B)$ and $(B \Rightarrow A)$" is equivalent to "$A \Leftrightarrow B$."

3. Prove each of the following using truth tables:

   (a) not(A or B) $\Leftrightarrow$ ((not A) and (not B))

   (b) not(A and B) $\Leftrightarrow$ ((not A) or (not B)) [the statements in (a) and (b) are called **deMorgan's laws**.]

   (c) $(A \Rightarrow B) \Leftrightarrow$ (not $B \Rightarrow$ not A) [This is called the **contrapositive**.]

   (d) ((A or B) $\Rightarrow C$) $\Leftrightarrow$ ($(A \Rightarrow C)$ and $(B \Rightarrow C)$)

   (e) ($A \Rightarrow$ (B and C)) $\Leftrightarrow$ ($(A \Rightarrow B)$ and $(A \Rightarrow C)$)

   (f) not(not A) $\Leftrightarrow A$

   (g) Discuss the importance of parentheses in the statements in this exercise. In part e), for instance, is "$A \Rightarrow$ (B and C)" the same as either "$(A \Rightarrow B)$ and C" or "$A \Rightarrow B$ and C"?

4. (a) Can "$A \Rightarrow B$" and "$A \Rightarrow$ (not B)" ever both be true?

   (b) Can "$A \Rightarrow B$" and "(not A) $\Rightarrow B$" ever both be true?

5. How many lines are there in a truth table that involves n statements?

6. (a) Construct a truth table to show that "$A \Rightarrow (B \Rightarrow C)$" is equivalent to "($A$ and B) $\Rightarrow C$."

   (b) Show that $(A \Leftrightarrow B) \Leftrightarrow$ (not $A \Leftrightarrow$ not B).

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[5] Exercises and Examples will be referred to by numbers indicating the section in which they appear. Exercise 1.4.2, for example, is the second exercise in Section 1.4; Example 18.2.1 is the first example in Section 18.2.

(c) Discuss why there is a "possible conflict in meaning" in the interpretation of "$\Leftrightarrow$".

(d) Show that "$A \Rightarrow (B \Rightarrow C)$" is equivalent to "($A$ and B) $\Rightarrow C$." by negating both statements and simplifying the results.

7. Show that the following are *not* true.

   (a) $(A \Rightarrow B) \Rightarrow (B \Rightarrow A)$.

   (b) (($A \Rightarrow B$) and B) $\Rightarrow A$.

   (c) ((A or B) and B) $\Rightarrow A$.

   (d) ((A or B) and B) $\Rightarrow$ not A.

8. In this section we have referred to "not (A and not B)" as a statement. Where is the verb in this expression?

1.5 PROOF BY CONTRADICTION

The equivalence in the last table in Section 1.4 is the basis for a technique called **proof by contradiction**, which is taken in practice as:

$$(A \Rightarrow B) \Leftrightarrow ((A \text{ and not } B) \text{ is false}).$$

You will verify this in Exercise 1.5.1. A proof by contradiction usually begins with the phrase "Suppose B does not hold ...," or something like it, and ends with "This is a contradiction."[6] Such proofs are also called "indirect." Because this technique is so useful, the negation of complex statements is an important skill. The most famous proof by contradiction is probably Euclid's proof that there are infinitely many prime numbers:

> Suppose there were only finitely many prime numbers. Then we could make a list of them all: $p_1, p_2, \ldots, p_n$. Consider the number $p = (p_1 \times p_2 \times \cdots \times p_n) + 1$. Observe that p is not divisible by any of the listed primes. But every number larger than 1 is divisible by some prime. This contradicts the assumption that our list contains *all* the primes, and the proof is finished.

[6] It is sometimes difficult to tell the difference between a proof by contrapositive (Exercise 1.4.3.c) and a proof by contradiction. A proof that ends *specifically* with the statement "not A" is a proof by contrapositive, while the contradiction in a proof by contradiction can arise from a number of sources. If you start out to do a proof by contradiction ("assume A and not B") and end with the contradiction "A is both true and false," your proof probably can be constructed as a contrapositive.

In this proof, the hypothesis (A) consists of a collection of statements about the factoring of natural numbers (which go unspecified here, but which would be clear in the context of a course in number theory). The conclusion (B) is "There are infinitely many primes." Euclid showed that "Certain statements about the factoring of natural numbers" and the statement "There are only finitely many primes" cannot both be true, that is, "A and not B" is false.

EXERCISES 1.5

1. Verify proof by contradiction: $(A \Rightarrow B) \Leftrightarrow ((A$ and not $B)$ is false$)$.

2. In Euclid's proof, is the number $p = (p_1 \times p_2 \times \cdots \times p_n) + 1$ necessarily prime itself? Compute p using the first few primes. Is p always prime?

1.6 CAUTION! THE SIREN SONG OF CONTRADICTION

There is temptation to use proof by contradiction far too often. Indirect proofs can be very attractive since they tend to be short and sometimes give us results almost by magic. But this comes at a cost. Magic is an activity in which what is really going on is *concealed* as much as possible. Our goal in writing a proof is to *reveal* as much as possible. A proof that seems to work by magic doesn't teach us much, and we should avoid such things whenever we can. This advice is given not only to ensure that we practice a variety of techniques (though that would be reason enough). There are schools of mathematics and philosophy in which proof by contradiction is not accepted, and such objections should not be dismissed lightly. Proof by contradiction rests on the assumption that a statement *must* be either true or false (in fact, this is our definition of "statement"). In life, there seem to be meaningful assertions that are neither true nor false ("Nice day, eh?")[7]. That there are only two truth values is called the "law of the excluded middle," which has been an assumption of Western logic for millennia. Is it true? Who knows? It is one of our axioms.

When should we use proof by contradiction, then? There is no single answer to this, only guidelines. A direct proof is usually preferable to an

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[7] One shouldn't conclude from this example that the issue is insignificant. Maybe statements should not be classified "true" and "false" or even as "true," "false," and "matters of opinion." It might be that "true," "false," and "it is impossible to tell" are the correct categories. This is a serious philosophical issue.

indirect proof (Euclid's proof can be reworded in such a way to make it direct, but its role in history entitles it to be left alone). The best hint that a proof by contradiction is appropriate is that the negation of the conclusion carries more information than the conclusion itself. In Euclid's proof, the negation of the conclusion (the assumption that the set of primes is finite) allows us to do something very useful: Make a list of them. Once we have specific names for the elements of this set, we can do arithmetic with them.

1.7 DISJOINED CONCLUSIONS

Consider the expression "$A \Rightarrow (B \text{ or } C)$." If a rule says "If it rains you must either have an umbrella or a raincoat," how could you defend yourself against the accusation "It is raining but you have no umbrella"? You could say "But I have my raincoat." The rule can be interpreted: "If it rains and you *don't* have an umbrella, you *must* have a raincoat" (or "If it rains and you don't have a raincoat, you must have an umbrella"). It seems that "$A \Rightarrow (B \text{ or } C)$" is equivalent to "$(A \text{ and not } B) \Rightarrow C$." It also seems that "$A \Rightarrow (B \text{ or } C)$" is equivalent to "$(A \text{ and not } C) \Rightarrow B$."

The truth table below checks the first of these (you will show the second equivalence in Exercise 1.7.1). The fifth and last columns of this table are the same, and so the two statements are equivalent. Notice that, since "$A \Rightarrow (B \text{ or } C)$" is equivalent both to "$(A \text{ and not } B) \Rightarrow C$" and to "$(A \text{ and not } C) \Rightarrow B$," *we do not need to establish both of the latter two statements to prove the first.* We can work with whichever looks the most promising, a decision we would make based on whether "not B" or "not C" gives us the most useful information.

A	B	C	B or C	$A \Rightarrow (B \text{ or } C)$	not B	A and not B	$(A \text{ and not } B) \Rightarrow C$
T	T	T	T	T	F	F	T
T	T	F	T	T	F	F	T
T	F	T	T	T	T	T	T
T	F	F	F	F	T	T	F
F	T	T	T	T	F	F	T
F	T	F	T	T	F	F	T
F	F	T	T	T	T	F	T
F	F	F	F	T	T	F	T

EXERCISES 1.7

1. Show that $(A \Rightarrow (B \text{ or } C)) \Leftrightarrow ((A \text{ and not } C) \Rightarrow B)$.

1.8 PROOF BY CASES—DISJOINED HYPOTHESES

Consider the expression "$(A \text{ or } B) \Rightarrow C$." You showed in Exercise 1.4.3.d that this is equivalent to "$(A \Rightarrow C)$ and $(B \Rightarrow C)$." This equivalence provides us with a very useful technique called **proof by cases**. Suppose we wish to show that it is always true that $x \leq |x|$. The definition of the absolute value changes depending on whether x is negative or not, and so we may rephrase the problem: "If x is either negative or nonnegative, then $x \leq |x|$." We've changed the hypothesis from "x is a number" to "$x < 0$ or $x \geq 0$." The proof can be done like this:

> **Case 1**: If $x < 0$, then $|x| = -x$, which is positive, and so $x < 0 < -x = |x|$ and $x \leq |x|$.
>
> **Case 2**: If $x \geq 0$, then $x = |x|$, and so $x \leq |x|$.
>
> In either case, $x \leq |x|$.

There are two important rules to keep in mind while constructing a proof by cases: First, the cases *must exhaust all possibilities*. In the above proof, at least one case must apply to any value of x we might chose. This proof would not be valid had we said "If $x < 0$ then $x \leq |x|$; if $x > 0$ then $x \leq |x|$," since we would not have established the result for $x = 0$. Second, all cases *must lead to the conclusion of the theorem*. It is common, but a bit careless, to say something like "If $x < 0$ then $x < |x|$; if $x \geq 0$ then $x = |x|$, consequently $x \leq |x|$."

Dividing a proof into cases is somewhat of an art. Sometimes the cases can be selected in more than one way so it is not clear what the cases should be, and sometimes it does not even become clear until late in a proof that the hypothesis contains a disjunction. When you have finished a proof by cases, you should examine it to see that the above two conditions are met and look to see whether part of your proof might be more general than you thought. We might have divided this proof into three cases: "If $x < 0$ then $x < 0 < |x|$, and so $x \leq |x|$; if $x > 0$ then $x = |x|$, and so $x \leq |x|$; if $x = 0$ then $x = 0 = |x|$, and so $x \leq |x|$; consequently $x \leq |x|$ for all x." The third case is resolved in the same way as the second, and so it was not necessary to consider it separately. This proof is not wrong, it is just not as good as it might be.

EXERCISES 1.8

1. (a) Show that $((A \Rightarrow C) \text{ and } (B \Rightarrow D)) \Rightarrow ((A \text{ or } B) \Rightarrow (C \text{ or } D))$.

   (b) How does (a) reflect on our comments about proof by cases?

   (c) Show that the converse of the implication in (a) does not hold. (You should first consider carefully what the converse is!)

2. If n is a natural number with $n > 3$, show that the expression $\frac{n!}{3!(n-3)!}$ is alway a natural number. (Hint: n must be either a multiple of 3, one more than a multiple of three, or two more than a multiple of 3.)

1.9 OPEN STATEMENTS AND QUANTIFIERS

How does an expression like "$x^2 + 2x + 1 = 4$" fit into our discussion? Technically speaking, it isn't grammatically correct (the symbol = indicates that the *numbers* on either side of it have the same value, but "$x^2 + 2x + 1$" is not a number), and so in the strictest sense it is not a statement at all. We understand, though, that x is a place holder (or **variable**). When we insert a number in its place, this expression becomes a statement.

Such expressions are called **open statements**, and they present special difficulties. An open statement might be true regardless of the value of the variable, like "$x^2 + 2x + 1 = (x + 1)^2$." It might be true for some values of the variable but not for others, like "$x^2 + 2x + 1 = 4$." Finally, it might be false for every value of the variable, as in "x is a real number and $x^2 = -1$."

We can't say whether an open statement is true or false until we specify the values of the variable for which our claim is to be made. This is called **quantification**. There are only two quantifiers: the **universal quantifier**, denoted $\forall$, and read "for each," "for all," "for every," or "whenever," and the **existential quantifier**, read "for some" or "there exists" and denoted $\exists$ (we *will* use these two symbols). Distinctions among the various "pronunciations" of the symbols are largely stylistic—use the one that sounds best. The phrase "such that" (denoted $\ni$) usually accompanies the existential quantifier for purely grammatical reasons. We write "$\exists x \ni (x > 3)$" and say "there exists x such that $x > 3$." The symbol $\ni$ serves no mathematical purpose, and it is best to consider it part of the quantification of the variable rather than part of the open statement.

A **quantified statement** consists of a list of variables with their quantifiers, followed by an open statement. We will call this **standard form.** The following quantified statements are true:

$$\forall x(\sin^2 x + \cos^2 x = 1)$$
$$\exists x \ni (x^2 + 2x + 1 = 4)$$
$$\exists x \ni (x \text{ is a real number and } x^2 = 3)$$
$$\forall x \exists y \ni (y^3 = x),$$

while these are false:

$$\forall x(\sin x + \cos x = 1)$$
$$\forall x(x^2 + 2x + 1 = 3)$$
$$\exists x \ni (x \text{ is a real number and } x^2 = -1)$$
$$\forall x \exists y \ni (y \text{ is a real number and } y^2 = x).$$

To make sense of quantified statements, we must assume some universe from which the values of the variables are to be chosen. *Unless we say otherwise, this will always be the set of real numbers.* The first statement above really should be written $\forall x(x \in \mathbf{R} \Rightarrow \sin^2 x + \cos^2 x = 1)$ or $\forall x \in \mathbf{R}\ (\sin^2 x + \cos^2 x = 1)$, but we usually dispense with explicit mention of the universe unless it is of special importance.

In ordinary language, we often fail to state quantifiers explicitly. For instance, if we say, "It rains where I live," we (probably) mean this to be existentially quantified: $\exists d \in \{\mathit{days}\} \ni$ (It rains on day d where I live). On the other hand, we probably would intend the statement "People should treat each other well" to be universally quantified. While this should not be a problem in mathematics, there is still confusion. If we say "Show that a symmetric matrix has real eigenvalues," the statement is meant to be *universally* quantified. An example (a calculation involving a specific matrix) does not prove this statement. Unstated quantifiers should always be taken to be universal.

If an open statement has more than one variable, the order in which they appear in the standard form is very important. The definition from calculus of the statement "The function f is continuous at the point a" is:

$$\forall \varepsilon > 0 \exists \delta > 0 \ni \forall x(|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon).$$

Writing this statement with the first two variables reversed, like this:

$$\exists \delta > 0 \ni \forall \varepsilon > 0 \forall x(|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon),$$

changes its meaning dramatically. It is left to the reader to decide what, if anything, the latter statement says about the function f.

In the definition of continuity, δ may change if ε changes. Generally, an existentially quantified variable may depend upon the variables appearing *before* it in the list. The definition of continuity would be more clear if we wrote it like this:

$$\forall \varepsilon > 0 \exists \delta(\varepsilon) > 0 \ni \forall x(|x - a| < \delta(\varepsilon) \Rightarrow |f(x) - f(a)| < \varepsilon).$$

If we had been careful to do this, it would have been clear that the two statements above are different since δ may depend on ε in the first but not in the second. Quantified statements are not often written this way, though.

The way a statement is quantified will guide us in constructing its proof. The basic methods for proving universally and existentially quantified statements are very different. It is extremely important to determine which sort of statement we are considering before we begin a proof, and we must always take the time to pick out any unstated quantifiers.

EXERCISES 1.9

1. In which of the examples of quantified statements given in this section is the universe important? (When could a change in the universe change the truth value?)

2. Characterize the functions for which

   → horizontal lines only, why
   → continuity @ a with the

   $$\exists \delta > 0 \ni \forall \varepsilon > 0 \forall x(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon).$$

   → has to be a delta that works, no matter what epsilon is.

3. Convince yourself of the truth values of the examples of quantified statements given in this section (we will not have the techniques to *prove* these statements until later).

4. (a) Consider the effect on a quantified statement of changing the quantifiers. What happens in the examples given if each $\forall$ is changed to a $\exists$ and vice versa? Are any of them still true? Does one that is true necessarily become false?

   (b) Consider the effect on a quantified statement of changing the quantifiers *and* negating the open statement.

1.10 PROVING UNIVERSAL STATEMENTS

The most direct way to prove a universally quantified statement would be to make a separate argument for each value of the variable. For instance, we may prove that $\forall x \in \{0, 1\}(x^2 = x)$ by checking that the statement is true if $x = 0$ (since $0^2 = 0$) and if $x = 1$ (since $1^2 = 1$). Notice that (i) the two "proofs" have no variables in them (an open statement becomes a statement when a value is inserted for the variable), and (ii) the proofs are different (though not much so in this case). This is not practical, if course, if the universe is larger, or if the problem is given in such a way that identifying all of the elements in the universe might be as difficult as doing the rest of the proof. To overcome this, we will construct proofs that consist of open statements (rather than statements), being careful that each step is valid for each value of the variable. If we can't construct a proof that is valid for all values of the variable at once, we might resort to a proof by cases. Since we can't keep in mind *every* value of the

quantified variable, we will begin such a proof by giving a name to *one* element of the universe. We then must be sure that our argument is valid for *that particular* element.

Here is a simple example. Let us prove: "*If* $x > 1$, *then* $x^2 > x$." Since there is no explicit quantifier in this statement, we assume it is intended to be true for *all* x greater than 1. We begin the proof by naming the element we will be discussing:

> Let $x > 1$. (sometimes we say "Let $x > 1$ be given."[8])
> Then it is also true that $x > 0$.
> So $(x)(x) > (x)(1)$.
> That is, $x^2 > x$.

We have used the fact that the ordering on the real numbers is transitive (in order to say that $x > 1$ and $1 > 0$ imply $x > 0$), and the fact that multiplying both sides of an inequality by a positive number does not change the inequality (both of these properties of real numbers will be proved in Theorem 4.12). Though we have made some assumptions about the ordering of the real numbers, the only property of x itself we have used is that it is greater than 1.

Here is another "example." I will "prove" that *If x is positive and $y < x$, then y is positive*(!). Here both variables are universally quantified, and so the proof begins like this:

> Let $x > 0$.
> Let $y < x$.
> Since $x > 0, x^2 > x$.
> So $x(x-1) = x^2 - x > 0$.
> Since $x > 0$ and $x(x-1) > 0$, we must have $x - 1 > 0$.
> This argument can be repeated to show $x - 2 > 0, x - 3 > 0, \ldots$.
> No matter what x is, these numbers go to $-\infty$, so one of them
> (say $x - k$) is smaller than y.
> So $y > x - k > 0$, and $y > 0$.

This proof is not valid (aside from the fact that the statement is silly), because it includes a statement that is not true for all stated values of x (which statement?). On the other hand, the proof did start correctly (whatever small comfort that might be) and *some* of the statements *are* valid inferences from those above them (which ones?).

---

[8] It might seem that "Let x is greater than 1" doesn't make much sense. This should be read "Let x *be* greater than 1." The second version should be read: "Let x—which is greater than 1—be given." The phrase "be given" reminds us that we do not get to pick the value of x ourselves. We know only, in this case, that it is greater than 1.

EXERCISES 1.10

1. What is wrong with the "proof" at the end of the section? Find the first statement in it that is not valid. For which values of which variable (if any) is that particular statement true? Is the proof valid if the statement of the problem is adjusted to include only those particular values of the variable? Decide whether each statement in the "proof" follows from those above it.

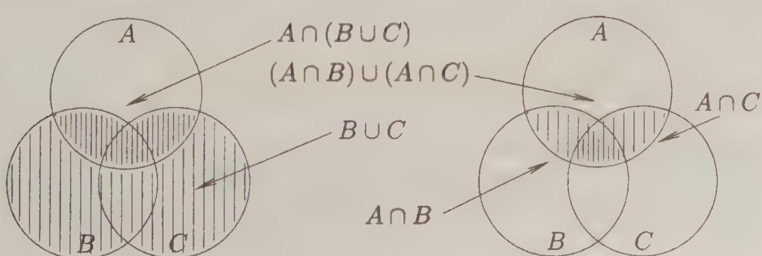
1.11 PROVING EXISTENTIAL STATEMENTS

This is easier to describe than the previous problem, but not always to carry out. The "technique" for proving existentially quantified statements can be summed up in one sentence (whose sweeping generality is in distinct contrast to its lack of practical advice!):

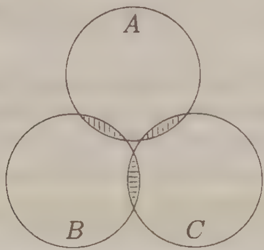
The best way to prove that something exists is to find one.

To prove that $\exists x \ni (x^2 + 2x + 1 = 4)$, we need only observe that the open statement becomes true when we set $x = 1$. Notice that we do not need to find *all* values of x for which the equation holds to show that the existential statement is true. Giving an example is not the only way to prove an existentially quantified statement. Sometimes it is possible to show that the assumption that something does not exist leads to a contradiction. But we should *always* look for an example first.

It is important to remember that, although we can (and should) prove existentially quantified statements by finding an example, a universally quantified statement *can't* be proved in this way. Observing that the inequality holds for $x = 3$ does not prove that $\forall x \in \mathbf{R}(x^2+2x+1 \geq 0)$. Pictures, such as Venn diagrams, are *examples* (they are sets of points) and so cannot be used to prove universal statements. They can give us ideas for proofs, though, and can provide examples for existential statements. For instance, here are Venn diagrams that *suggest* that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$:



These diagrams might seem to give compelling evidence, but they don't *prove* the result, because they are just one example (we do have some idea, of course, that such a Venn diagram is a very general representation of the possible relationships among three sets, but sorting this out is not the issue now.). On the other hand, the Venn diagram below *proves* that *it is possible to have $A \cap B \cap C = \emptyset$ even though none of $A \cap B$, $A \cap C$, nor $B \cap C$ is empty.* Notice that the quantifier associated with the picture above is universal, while the statement being described in the picture below has an existential quantifier ("It is possible").



1.12 NEGATING A QUANTIFIED STATEMENT

There is only one technique for negating quantified statements. Even in ordinary (careful) language, the opposite of "This always happens" is "There is one instance when it does not happen" (but *not* "This never happens"). On the other hand, the opposite of "This happens at least once" is "This never happens." The negation of quantified statements is relatively simple if they are written in standard form: Change each quantifier to the other one (remember there are only two), and negate the open statement. The statement $\forall x \in \mathbf{R}(x^2 + 2x + 1 \geq 0)$ is negated: $\exists x \in \mathbf{R} \ni (x^2 + 2x + 1 < 0)$. This works for statements with any number of variables. The negation of $\exists x \ni \forall y (x^2 = y)$ is $\forall x \exists y \ni (x^2 \neq y)$. (Which of these is true?) The statement "The function f is continuous everywhere" would be written (note that "everywhere" is a universal quantifier):

$$\forall a \forall \varepsilon > 0 \exists \delta > 0 \ni \forall x (|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon).$$

The negation of this is then:

$$\exists a \exists \varepsilon > 0 \ni \forall \delta > 0 \exists x \ni (|x - a| < \delta \text{ and } |f(x) - f(a)| \geq \varepsilon).$$

[Recall that (not $(A \Rightarrow B)$) $\Leftrightarrow$ (A and not B).] Changing the quantifiers can also change which variables depend on which others. In the first statement above, δ depends on ε and a. In the second statement, it doesn't.

EXERCISES 1.12

1. Which of the statements $\exists x \ni \forall y(x^2 = y)$ or $\forall x \exists y \ni (x^2 \neq y)$ is true?

2. Translate each of the following into a quantified statement in standard form, write its symbolic negation, and then state its negation in words:

   (a) Everybody loves somebody sometime.

   (b) You can fool all of the people some of the time.

   (c) You can't teach an old dog new tricks.

   (d) When it rains, it pours.

   (e) Things have never been more like they are right now.

   (f) If we don't hang together, surely we shall all hang separately.

   (g) Out of sight, out of mind.

3. (a) Define (in words) the phrase "The function $f : A \to B$ is one-to-one." (If you've never encountered this phrase before, skip this problem.)

   (b) Carefully state (in standard form) the definition of the phrase "The function $f : A \to B$ is one-to-one."

   (c) Carefully state (in standard form) the definition of the phrase "The function $f : A \to B$ is *not* one-to-one."

   (d) Define (in words) the phrase "The function $f : A \to B$ is *not* one-to-one."

   (e) Repeat (a) through (d) with "one-to-one" replaced by "onto."

1.13 THE FORWARD-BACKWARD METHOD

We have seen that in certain circumstances we can tell, just by looking at the type of problem we are trying to solve, where a proof should begin and where it should end. This idea is the beginning of a powerful technique called the **forward-backward method** (the name seems to have been coined by Daniel Solow in his book *How to Read and Do Proofs*—see the references). The forward-backward method is the mathematical equivalent of building a chain from both ends at once. As builders of proofs rather than chains, we enjoy some advantages over blacksmiths. The collections of steps and reasons from which we may choose are fairly small, and each step can be linked to only a few others, often in only one way. Thinking in this way, the forward-backward method can be boiled down to some simple advice:

(1) Every statement begins somewhere and ends somewhere: An equality begins on one side and ends on the other; an implication begins with the hypotheses and ends with the conclusion; and so on.

(2) We want proofs to get us from one place to another. Think of the things you know that *begin* where you *are* (statements whose hypotheses match the hypotheses of the problem). Now think of the things you know that *end* where you *want to go* (statements whose conclusions match the conclusions of the problem).

(3) Put the things you thought of in (2) in place in the chain. The ends of the chain should be closer together in some sense than they were before. Essentially, your hypotheses and conclusions have changed. Now go back to step (2) and start again.

Let's do a simple proof: *If A and B are sets and $A \subseteq B$, then $C(B) \subseteq C(A)$.* [$C(A)$ is the complement of A.] Our goal is to show that one set is a subset of another [$C(B) \subseteq C(A)$], so we must show that every element of $C(B)$ is also an element of $C(A)$. This indicates a universal quantifier, and so we already know what the first and last steps of our proof must be. (Proofs involving sets will be discussed in more detail in Section 1.15.) We will indicate a proof by the forward-backward method with the following structure.

▽▽

Let $x \in C(B)$.

⋆ ⋆ ⋆

Then $x \in C(A)$.

△△

We know the definition of the complement of a set, and since we know nothing else about A and B, it seems that this definition will have to come into play. We can insert another step at the top and one at the bottom (we will indicate new lines with the symbol $\hookrightarrow$):

▽▽

Let $x \in C(B)$.

$\hookrightarrow$ Then $x \notin B$.

⋆ ⋆ ⋆

$\hookrightarrow$ $x \notin A$.

Then $x \in C(A)$.

△△

We have taken advantage of the fact that the definition of the complement

of B *begins* at "$x \in C(B)$" and *ends* at "$x \notin B$." [Since it is a definition, it also begins at "$x \notin A$" and ends at "$x \in C(A)$."] We haven't written "Then $x \notin A$" at the bottom, since we aren't yet sure that it will follow from things above it (we hope it will, but this will not be a proof until all the pieces are in place). This is, of course, neither a very long nor a very difficult proof. We should observe at this stage only that the pieces of the chain at the beginning and at the end are valid bits of reasoning. Our problem now has changed. Now we don't have to worry about whether the first statement in our proof implies the last, but only whether the last statement at the top ($x \notin B$) implies the first statement at the bottom ($x \notin A$). We do know that $A \subseteq B$. If x were an element of A, then it would have to be an element of B, and this is not the case. This is a small proof by contradiction, which we now insert, and we are done:

▽▽▽

Let $x \in C(B)$.

Then $x \notin B$.

$\hookrightarrow$If x were and element of A, it would have to be an element of B, and this is not the case.

$x \notin A$.

Then $x \in C(A)$.

△△△

We can now view the whole proof, inserting the reasons for each step on the right (this will be left to you for most of the book).

▽▽▽

Let $x \in C(B)$.	*This is where the proof must start.*
Then $x \notin B$.	*Definition of the complement.*
If x were an element of A, it would have to be an element of B, and this is not the case.	*Follows from the line above, since $A \subseteq B$.*
$x \notin A$.	*From the two lines above.*
Then $x \in C(A)$.	*This is where the proof must end.*

△△△

We don't always have to alternate between backward and forward steps. (In practice, most proofs are constructed primarily of backward steps.) Notice that the last step we inserted was not simply a definition or theorem. We got it by considering where we stood in the proof and the information available to us. The forward-backward method usually is not sufficient to do an entire proof.

1.14 A BIGGER AND BETTER EXAMPLE

We have just made quite a big deal of a very simple proof. Let us try the method on something substantial. We will prove a result from calculus:

If the function f is differentiable at a point a,
then it is continuous there.

This is a significant result, and the proof is not at all easy. Adding to the difficulty, we will discover that it is an "existence" proof, so we will need to find an example of some sort. Existence steps are the most difficult part of any proof, and we will come to find, sadly, that they are where the forward-backward method is least helpful. However, the forward-backward method will help make it clear where existence steps occur in a proof, allowing us to focus our attention where it is most needed.

You may have seen this before, though probably not in this much detail. Don't be concerned with every step. What is important now is the way the proof is constructed. We know how it begins and ends:

▽▽

f is differentiable at the point a.

⋆ ⋆ ⋆

f is continuous at the point a.

△△

The words "differentiable" and "continuous" have definitions:

▽▽

f is differentiable at the point a.

$\hookrightarrow \lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ exists (we may call it L).

⋆ ⋆ ⋆

$\hookrightarrow \lim_{x\to a} f(x) = f(a)$.

f is continuous at the point a.

△△

Our chain now has two links at each end. The statement we have inserted at the bottom—$\lim_{x\to a} f(x) = f(a)$—is defined:

$$\forall\varepsilon > 0 \exists\delta > 0 \ni \forall x(0 < |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon).$$

This is *universally* quantified (whatever else goes on in the proof, we must show that something happens "$\forall\varepsilon > 0$"). This tells us what to do next:

▽▽▽

f is differentiable at the point a.

$\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists (we may call it L).

$\hookrightarrow$ Let $\varepsilon > 0$ be given.

★ ★ ★

$\lim_{x \to a} f(x) = f(a)$.

f is continuous at the point a.

△△△

We need to show that something happens for *this particular* ε:

▽▽▽

f is differentiable at the point a.

$\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists (we may call it L).

Let $\varepsilon > 0$ be given.

★ ★ ★

$\hookrightarrow \exists \delta > 0 \ni \forall x (0 < |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon)$.

$\lim_{x \to a} f(x) = f(a)$.

f is continuous at the point a.

△△△

We have found that our goal in the proof has become to show that this δ exists, meaning that we will have to give an example of such a δ. Let us turn our attention back to the forward part of the proof. The definition of the limit there tells us something about this ε:

▽▽▽

f is differentiable at the point a.

$\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists (we may call it L).

Let $\varepsilon > 0$ be given.

$\hookrightarrow \exists \delta_1 > 0 \ni \forall x \left(0 < |x - a| < \delta_1 \Rightarrow \left| \dfrac{f(x) - f(a)}{x - a} - L \right| < \varepsilon \right)$.

★ ★ ★

$\exists \delta > 0 \ni \forall x (0 < |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon)$.

$\lim_{x \to a} f(x) = f(a)$.

f is continuous at the point a.

△△△

Here we have used a little foresight. We don't know whether the δ that the definition of the limit gives us here is the one we're looking for, and so we have given this one a different name.

Now we have reached the point where we will need to do some creative work. The problem gets a little harder, and you can consider the lesson of the forward-backward method to be over (the next few paragraphs can be safely skipped at this time). The line we've just inserted looks a little like the one we want to reach (at least it has some of the same pieces). Perhaps we can manipulate them to make them look more alike. Look at the step we can reach from the beginning:

$$\exists \delta_1 > 0 \ni \forall x \left(0 < |x-a| < \delta_1 \Rightarrow \left|\frac{f(x)-f(a)}{x-a} - L\right| < \varepsilon\right)$$

and where we need to go:

$$\exists \delta > 0 \ni \forall x (0 < |x-a| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon).$$

We need to make the first of these look more like the second. The expression $f(x)-f(a)$ appears in both, but it is sitting all by itself in the second line. Let's put it by itself in the first. With a little algebra, we can change the conclusion in the first line to: $|f(x)-f(a)-L(x-a)| < \varepsilon|x-a|$. We will insert this into our proof in a moment, but it still is not quite what we want, since $|f(x)-f(a)|$ is not yet by itself. Some free association is needed. Is there a way to relate an expression like $|X-Y|$ to $|X|$ and $|Y|$? There is. It's a version of the Triangle inequality: $|X|-|Y| \leq |X-Y|$. Letting $f(x)-f(a)$ be X and $L(x-a)$ be Y, and doing a little algebra, we find $|f(x)-f(a)| < \varepsilon|x-a| + |L||x-a|$. With bit more algebra, we can insert:

▽▽

f is differentiable at the point a.

$\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ exists (we may call it L).

Let $\varepsilon > 0$ be given.

$\exists \delta_1 > 0 \ni \forall x \left(0 < |x-a| < \delta_1 \Rightarrow \left|\dfrac{f(x)-f(a)}{x-a} - L\right| < \varepsilon\right)$.

• ⟶ $\exists \delta_1 > 0 \ni \forall x (0 < |x-a| < \delta_1 \Rightarrow |f(x)-f(a)| < (\varepsilon + |L|)|x-a|)$.•

⋆ ⋆ ⋆

$\exists \delta > 0 \ni \forall x (0 < |x-a| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon)$.

$\lim\limits_{x\to a} f(x) = f(a)$.

f is continuous at the point a.

△△

Now we are getting very close. We have picked δ_1 in such a way as to make $|f(x) - f(a)|$ less than *something*, but we need it to be less than ε. The right side of the inequality marked by •• will be less than ε if we can make $|x - a|$ less than $\varepsilon/(\varepsilon + |L|)$. Let us set $\delta_2 = \varepsilon/(\varepsilon + |L|)$. Then if $|x-a| < \delta_2$, the right side of the •• inequality will be less than ε. But we must be cautious. In order for the •• inequality to hold at all, we must have $|x - a| < \delta_1$. We need *both* of these things to happen at once, and we can do that with a careful choice of δ:

▽▽

f is differentiable at the point a.

$\lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a}$ exists (we may call it L).

Let $\varepsilon > 0$ be given.

$\exists \delta_1 > 0 \ni \forall x \left(0 < |x - a| < \delta_1 \Rightarrow \left|\dfrac{f(x) - f(a)}{x - a} - L\right| < \varepsilon\right).$

$\exists \delta_1 > 0 \ni \forall x (0 < |x - a| < \delta_1 \Rightarrow |f(x) - f(a)| < (\varepsilon + |L|)|x - a|).$

$\hookrightarrow$ Let $\delta_2 = \varepsilon/(\varepsilon + L)$.

$\hookrightarrow$ If $0 < |x - a| < \delta_2$, then $(\varepsilon + |L|)|x - a| < (\varepsilon + |L|)\left(\dfrac{\varepsilon}{\varepsilon + |L|}\right) = \varepsilon.$

$\hookrightarrow$ Let δ be the smaller of δ_1 and δ_2.

$\hookrightarrow$ If $0 < |x - a| < \delta$, then $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$.

$\hookrightarrow$ If $0 < |x - a| < \delta$, then $|f(x) - f(a)| < (\varepsilon + |L|)|x - a| < \varepsilon$.

$\exists \delta > 0 \ni \forall x (0 < |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon).$

$\lim\limits_{x\to a} f(x) = f(a).$

f is continuous at the point a.

△△

This difficult proof is finished, and since the second, fourth, and fifth of the steps we have just inserted serve only to explain the first and the third, note that *only two of the steps required any creative activity*! The great value of the forward-backward method is that it allows us to deal with those parts of a proof that are merely structural in a routine way, leaving us to focus our attention on those issues that really require work. We will examine proofs in this way throughout the book. Though it will not always finish a problem for us, we will find that the forward-backward method is almost always useful in locating the real issues in a proof.

EXERCISES 1.14

1. Give a much briefer explanation (not a proof) of why the theorem in

this section is true. (The denominator of a certain fraction goes to 0. What does the numerator do?)

1.15 SETS

We need not dwell on the technical aspects of set theory here. We are already reasonably good at manipulating sets. We will consider now only how the ideas we've discussed in this chapter relate to proofs involving sets. We are generally concerned with only two possible relationships between sets—containment and equality—and four operations—union, intersection, complement, and relative complement (or difference).

CONTAINMENT: The statement "A is a subset of B" is defined: $\forall x(x \in A \Rightarrow x \in B)$. This is universally quantified, and so we know that the proof of a theorem in which we must show that $A \subseteq B$ should begin with the phrase "Let $x \in A$." This usually should be followed immediately by a statement of the definition of A.[9] The rest of the proof must consist entirely of statements that are true for any element of A, and it must end with "Then $x \in B$," which would usually be immediately preceded by the definition of B. There is a great temptation to look for shortcuts in such proofs, but *the number of set containment proofs that can be done properly any other way is so small that it's not usually worth considering other approaches.*

EQUALITY: Two sets are equal if each is a subset of the other. A proof that $A = B$ has two parts, one of which should begin "Let $x \in A$" and end "Then $x \in B$," the other beginning "Let $x \in B$" and ending "Then $x \in A$." *The number of set equality proofs that can be done properly any other way is so small that it's not usually worth considering other approaches.*

SET OPERATIONS: The definitions of the various set operations are:

$$x \in A \cup B \text{ if } ((x \in A) \text{ or } (x \in B)).$$

$$x \in A \cap B \text{ if } ((x \in A) \text{ and } (x \in B)).$$

$$x \in C(A) \text{ if } (\text{not}(x \in A)).$$

$$x \in A \backslash B \text{ if } ((x \in A) \text{ and not } (x \in B)).$$

---

[9] This helps remind us that the proof can rely only on the assumption that $x \in A$. *Note well* that in saying "Let $x \in A$" at this point in a proof we are *not* asserting that A actually has any elements. If such a proof is valid, it will be so if $A = \emptyset$.

These involve only the usual connectives, each of which we have considered. For instance, to prove that $x \in A \cup B$—a disjoined conclusion—we should begin "Suppose $x \notin A$" and conclude "Then $x \in B$" or vice versa.

EXERCISES 1.15

1. Suppose A, B, and C are sets. Do the following for each of these statements. First, construct a Venn diagram that indicates that the result is true, then prove the result, using the forward-backward method whenever you can.
   (a) $A \subseteq B \Leftrightarrow C(B) \subseteq C(A)$.
   (b) $C(A \cup B) = C(A) \cap C(B)$.
   (c) $C(A \cap B) = C(A) \cup C(B)$.
   (d) $A \subseteq (B \cap C) \Leftrightarrow (A \subseteq B)$ and $(A \subseteq C)$.
   (e) $(A \cup B) \subseteq C \Leftrightarrow (A \subseteq C)$ and $(B \subseteq C)$.
   (f) $A \subseteq (B \cup C) \Leftrightarrow (A \backslash C) \subseteq B$.
   (g) If $A \subseteq B$, then $B \backslash (B \backslash A) = A$.
   (h) Is the result in (g) true if $A \not\subseteq B$?

2. Use Venn diagrams to show that it is *not* true that
   (a) $(A \subseteq B) \Rightarrow (B \subseteq A)$.
   (b) $((A \subseteq B)$ and $(x \in B)) \Rightarrow (x \in A)$.
   (c) $((x \in A \cup B)$ and $(x \in B)) \Rightarrow (x \in A)$.
   (d) $((x \in A \cup B)$ and $(x \in B)) \Rightarrow (x \notin A)$.

3. Compare Exercises 1.4.3 and 1.7.1 with Exercise 1.15.1, and compare Exercise 1.4.7 with Exercise 1.15.2. Consider whether there is a relationship between symbolic logic and elementary set theory.

4. Let $S = \{x : x = 5n, n = 1, 2, \ldots\}$ and $T = \{x : x = 10n, n = 1, 2, \ldots\}$. Show *in detail* that $T \subseteq S$.

5. If S and T are sets, show that $S \backslash T = \emptyset$ if and only if $S \subseteq T$.

6. The set-theoretic equivalent of exclusive disjunction is called the **symmetric difference** of two sets and is given by $S \triangle T = (S \backslash T) \cup (T \backslash S)$.
   (a) Make a Venn diagram illustrating $S \triangle T$.
   (b) Show that $S \triangle T = (S \cup T) \backslash (S \cap T)$.
   (c) Explain why this is like exclusive disjunction.

7. Unions and intersections involving infinite collections of sets are defined as follows. Let $\{S_\alpha : \alpha \in \mathcal{A}\}$ be a collection of sets ($\mathcal{A}$ is an index set that can be of any size). Then

$$\bigcup_{\alpha\in\mathcal{A}} S_\alpha = \{x : \exists \alpha \in \mathcal{A} \ni (x \in S_\alpha)\}$$
$$\text{and} \quad \bigcap_{\alpha\in\mathcal{A}} S_\alpha = \{x : \forall \alpha \in \mathcal{A}(x \in S_\alpha)\}.$$

If the index set is the set of natural numbers, we write

$$\bigcup_{n=1}^{\infty} S_n \quad \text{or} \quad \bigcap_{n=1}^{\infty} S_n.$$

If the index set is known from the context of the problem, we can write

$$\bigcup_n S_n, \ \bigcup_\alpha S_\alpha, \ \bigcap_n S_n, \ \text{or} \ \bigcap_\alpha S_\alpha,$$

which mean "the union or intersection over all possible values of n or α." When we use a Greek letter for a subscript, we are making no statement about the size of the index set, while using a roman letter indicates a countable index set (the size of a set and what it means for a set to be countable are discussed in the next chapter).

(a) State the definitions of union and intersection in words.

(b) Show that the distributive laws hold in the following forms:

$$\bigcup_{\alpha\in\mathcal{A}}(T \cap S_\alpha) = T \cap \left(\bigcup_{\alpha\in\mathcal{A}} S_\alpha\right)$$
$$\text{and} \quad \bigcap_{\alpha\in\mathcal{A}}(T \cup S_\alpha) = T \cup \left(\bigcap_{\alpha\in\mathcal{A}} S_\alpha\right).$$

(c) Show that the union of a collection of sets is the smallest set that contains all the sets in the collection.[10]

(d) Show that the intersection of a collection of sets is the largest set that is contained in all the sets in the collection.

8. State and prove analogues of DeMorgan's laws for infinite collection of sets.

9. If a collection of sets is indexed with the natural numbers, we define the **limit superior** and **limit inferior** of the collection, respectively, as follows:

$$\limsup\{S_n\} = \bigcap_{k=1}^{\infty}\left(\bigcup_{n=k}^{\infty} S_n\right)$$
$$\text{and} \quad \liminf\{S_n\} = \bigcup_{k=1}^{\infty}\left(\bigcap_{n=k}^{\infty} S_n\right).$$

[10] One must be careful using the words "smallest" and "largest" in this context. The **smallest** set with a given property is that set (if any) that (i) has the property and (ii) is contained in any set having the property. The **largest** set with a given property is that set (if any) that (i) has the property, and (ii) *contains* any set having the property.

(a) Show that $\limsup\{S_n\} = \{x : x \in S_n$ for infinitely many $n\}$.

(b) Show that $\liminf\{S_n\} = \{x : x \in S_n$ for all but finitely many $n\}$.

(c) Show that $\liminf\{S_n\} \subseteq \limsup\{S_n\}$, and that they might not be equal.

(d) If $S_1 \subseteq S_2 \subseteq \ldots$, show that $\limsup\{S_n\} = \bigcup_{n=1}^{\infty} S_n$.

(e) If $S_1 \supseteq S_2 \supseteq \ldots$, show that $\liminf\{S_n\} = \bigcap_{n=1}^{\infty} S_n$.

(f) Give examples of collections where the identities in (d) and (e) fail.

10. (a) Show that a set can be an element of itself. (In some constructions of set theory, there is an axiom that prevents this. We will see why shortly.)

(b) Show that a set might not be an element of itself.

(c) Consider the collection of sets $R = \{S : S \notin S\}$. Show that *both* the assumptions $R \in R$ and $R \notin R$ lead to contradictions. If R were a set, one or the other of these statements would have to be true. We must conclude that R is *not a set*. (This is called "Russell's Paradox," after Bertrand Russell. Russell's discovery of this problem forced a radical change in the way mathematicians of the time viewed set theory.)

1.16 A GLOSSARY

Here are a few terms that will be used throughout the book. We have already said what **axioms**, **statements**, and **proofs** are.

A theorem consists of one or more statements (the **hypotheses**) that we intend to prove imply one or more other statements (the **conclusions**). We should view a theorem as a dynamic process that is not over until the proof is complete. Once we have proved a theorem, it becomes a (true) statement.

The structure of a theorem should help us decide which statements are hypotheses and which are conclusions. If this is unclear, we should rewrite the theorem (being careful not to change its meaning) in the form:

IF {the hypotheses} **THEN** {the conclusions}.

Owing to laziness and tradition, theorems are not always stated this way, but we should always be prepared to convert one to this form if necessary. We will indicate the end of a proof with the symbol ∎. Sometimes we put this symbol immediately after the statement of a theorem, where it means

either "We have already proved this" or "You will prove this yourself." *It is traditional in mathematics texts to print the statements of theorems in type that looks like this.* You have already seen this done from time to time in this chapter.

The **converse** of the theorem $A \Rightarrow B$ is the theorem $B \Rightarrow A$. The proof of the converse of a theorem is a separate issue from the proof of the original theorem. It is a common error to mistake the truth of a theorem for the truth of its converse. "I see what I eat" (If I eat it, then I see it) is different from "I eat what I see" (If I see it, then I eat it).

An **if and only if theorem** is a theorem in conjunction with its converse. The two parts must be proved separately.

A **definition** has the form of an if and only if theorem, where one of the parts (the one being defined) had no previous meaning. A definition is taken to be true, and can't be proved. In this way, definitions play much the same role as axioms. As we have done already, we will use boldface to indicate the term being defined, for instance: "A triangle is **equilateral** if and only if all its sides have the same length." Tradition again leads to bad habits, and definitions are usually written with "if" where they should have "if and only if." A definition always includes (if only implicitly) the connective "if and only if."

A **lemma** is a special sort of theorem that represents a portion of the proof of another theorem, pulled aside to be proved separately. Lemmas are pieces of our chains from the middle, which may be assembled (that is, proved) independently of the rest of the chain. Arranging a proof into lemmas is a bit of an art. It is usually done to improve the organization of a proof, but sometimes because a lemma is interesting in itself. In practice, it is almost always done after a proof is complete (when we notice that some part of a proof can stand on its own). Calling something a lemma indicates that it will be used primarily as part of another proof.

A **corollary** is also a theorem, usually stated just after another has been proved, whose proof is based mostly on the just-completed theorem. If we can prove that $n < 2^n$ *for all* n, the statement "$n - 1 < 2^n$ *for all* n" is a corollary. (The proof would start: "Note that $n - 1 < n$ and refer to the previous theorem.") Sometimes a corollary makes reference to some part of a preceding proof, rather than to the whole theorem, or to more than one theorem. In any event, calling something a corollary indicates that its proof will consist primarily of a reference to another theorem.

We will think of **functions** in a very simplistic way. A function f from

the set A to the set B will be denoted $f : A \to B$, where A is called the **domain** of f, and B is called the **range** (or sometimes the **codomain**[11]). We view the function itself as a rule by which we can, for each element of A, produce an element of B. We speak of the elements of A as "inputs," and the result of applying the rule f to an input x as an "output," which we denote $f(x)$.

1.17 PLAIN AS THE NOSE ON YOUR FACE?

A final note on the careful use of language. Words like "it is obvious" or "clearly" have no place in correct proofs. They are used all the time anyway, and they are used in this book. What they mean in this book is "I leave it to you to supply the details and am very sure that you can (but you should do so before moving on)." Stories abound of lengthy and intense discussions in class aimed at deciding whether some statement or another is indeed "obvious" (some of these stories are even true). If you see the humor in this, you're well on your way to becoming a mathematician. Here is a simple guideline for now: If you have to think about something at all—if you are even momentarily unsure that it is true—then it is not obvious. That $1+1=2$ is obvious. That $x=6$ is a solution to $2x^2+5x-102=0$ is *not* obvious.

EXERCISES 1.17

1. It has been suggested that one should avoid having a mathematician on a jury, because they have difficulty with the concept of "reasonable doubt." Discuss.

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[11] There is an important distinction between the two terms, but it won't become an issue for us until Chapter 8.

Chapter 2

Finite, Infinite, and Even Bigger

2.1 CARDINALITIES

When we count a set, we try to match its elements with the elements of some **initial segment** of the natural numbers: $\{1, 2, \ldots, n\}$. To be sure we've counted correctly, we should check that (i) each element of the set is associated with an element of the initial segment; (ii) each element of the initial segment is associated an element of the set; (iii) no element of the set is associated with more than one element of the initial segment; and (iv) no element of the initial segment is associated with more than one element of the set. Having done all this, we say that n is the number of elements in the set. Here we make these ideas precise.

DEFINITION 2.1: (a) A function $c : S \to T$ is a **one-to-one correspondence** if it has the following properties:

(i) $\forall x \in S \, \forall y \in S (x \neq y \Rightarrow c(x) \neq c(y))$ [that is, c is **one-to-one**],
and (ii) $\forall z \in T \, \exists x \in S \ni (c(x) = z)$ [that is, c is **onto**].

(b) If a one-to-one correspondence exists between two sets, we say that the sets are (or can be put) **in one-to-one correspondence**.

(c) A set is **finite** (and **contains n elements**) if it can be put in one-to-one correspondence with an initial segment $\{1, 2, \ldots n\}$.

(d) A set that is not finite is **infinite**.

You will show in Exercise 2.1.1 that this definition is not as directional as it seems, and that two sets that can each be put in one-to-one correspondence with a third set can be put in one-to-one correspondence with each other. This paves the way for the next two definitions.

DEFINITION 2.2: (a) Two sets **have the same cardinality** if they can be put in one-to-one correspondence with each other.

(b) If such sets are finite, we say they **have the same number of elements**.

The simple idea of associating sets this way in order to count their elements has surprising consequences, as we shall see. Sets that seem to be of very different sizes can have the same cardinality. For instance, the set of even natural numbers can be put in one-to-one correspondence with all the natural numbers, even though the set of all natural numbers seems to be bigger:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & \dots \end{array}$$

When we learned as children about the number 3, for instance, we were shown example after example of sets with three elements: three apples, three cats, two carrots and a bunny (things were simpler then). We came to understand that these sets had something in common that had nothing to do with their specific elements. That common property is what we learned to call "3." This is a difficult idea to make precise, but the following definitions tell part of the story.

DEFINITION 2.3: (a) A **cardinality** is the property common to a collection of sets that can be put in one-to-one correspondence with each other, but that is not shared by any set that can't be put in one-to-one correspondence with a set in the collection.

(b) If the sets in such a collection are finite, the cardinality is also said to be **finite**. A finite cardinality is called a **natural number.**[1]

Notice that we don't refer to "the set of *all* sets that can be put in one-to-one correspondence with each other," since such a collection is so large that it falls outside the realm of set theory (see Exercise 1.15.10).

EXERCISES 2.1

1. (a) In what way is Definition 2.1.a "directional"?

   (b) Show that (i) any set can be put in one-to-one correspondence with itself; (ii) if A can be put in one-to-one correspondence with B, then B can be put in one-to-one correspondence with A; and (iii) if A and B can both be put in one-to-one correspondence with C, then A can be put in one-to-one correspondence with B.

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[1] We are being a little free with our understanding of the natural numbers, and this is a rather circular definition. An alternate characterization of "finite" can be found in Exercises 2.1.4 and 2.3.7.

(c) Explain how (ii) serves to eliminate the directional quality of Definition 2.1.a.

2. (a) By considering the function $f(x) = \arctan x$, show that the set of real numbers has the same cardinality as the interval $(-\pi/2, \pi/2)$. ["Interval" hasn't been precisely defined yet—nor has the set of real numbers—but this should not be a problem here.]

(b) By considering the function $f(x) = \dfrac{x}{1+|x|}$, show that the set of real numbers has the same cardinality as the interval $(-1, 1)$.

(c) Show that any two nonempty open intervals have the same cardinality.

(d) Show that the set of real numbers has the same cardinality as any nonempty open interval.

(e) Show that the closed interval $[0, 1]$ has the same cardinality as the open interval $(0, 1)$. (You might not be able to do this by defining a function. You may want to delay this until you have considered Exercise 2.3.7.)

(f) Show that the set of real numbers has the same cardinality as any nonempty *closed* interval that does not consist of a single point. (You can do this whether you have solved (e) or not.)

3. (a) Let I be the set of decimals of the form $0.d_1d_2\ldots$. Construct a one-to-one function from I to $I \times I$.

(b) Find either an onto function from I to $I \times I$ or a one-to-one function for $I \times I$ to I.

(c) Do I and $I \times I$ have the same cardinality?[2]

4. (a) Let $f : A \to B$ be one-to-one and onto. Define $g : B \to A$ by saying $a = g(b) \Leftrightarrow b = f(a)$. Show that g is a function and that it is one-to-one and onto.

---

[2] This exercise can be interpreted as saying that the cardinality of the *inside of a square* $(I \times I)$ is the same as that of an *interval on the number line* (I), which should come as quite a surprise. But this is a trick question. You showed in (a) that the cardinality of I is *not larger* than that of $I \times I$. You showed in (b) that the cardinality of I is *not smaller* than that of $I \times I$. If the cardinality of A is both not smaller and not larger than that of B, do A and B have the same cardinality? The result saying that this is so is called the Cantor-Bernstein-Dedekind theorem. It took some of the greatest minds of the era to prove this!

(b) Show that, if $a \in A$, then $g(f(a)) = a$, and if $b \in B$, then $f(g(b)) = b$. Explain why g is called the **inverse function** of f (inverse functions are discussed further in Chapter 8).

(c) Examine your proof of (a) and carefully pick out which properties of f lead to which properties of g. (For instance, we know that f is one-to-one. Does this fact *alone* tell us anything about g?)

(d) If A is finite and $f : A \to B$ is one-to-one, show that the number of elements of $f(A) = \{y \in B : \exists x \in A \ni (y = f(x))\}$ is the same as the number of elements of A.

(e) Show that it is impossible to have a one-to-one correspondence between a finite set and one of its proper subsets. (Hint: Suppose that it *is* possible to have $C \subseteq B$, $C \neq B$, and a one-to-one function $f : B \to C$. Consider the set B having the *smallest* number of elements for which this happens.)

5. Here we generalize the results of Exercise 2.1.1. A **relation** on a set X is a set of ordered pairs of elements of X. A relation is often denoted by a symbol like $\approx$, and we write "$x \approx y$" (and say "x is related to y") to indicate that (x, y) is an element of the relation. The relation $\approx$ is called an **equivalence relation** on X if it has the properties:

(i) $x \approx x$ for all $x \in X$

(ii) if $x \approx y$ then $y \approx x$

and (iii) if $x \approx y$ and $y \approx z$ then $x \approx z$.

(a) Show that $=$ and $\leq$ are equivalence relations on the real numbers, but that $<$ is not. (For instance, show that the relation defined by saying $x \approx y$ if $x = y$ is an equivalence relation.)

(b) Is $\subseteq$ an equivalence relation on sets?

(c) Is ... is related to ... an equivalence relation on the set of people?

(d) Is ... is acquainted with ... an equivalence relation on the set of people?

(e) Let X be a set and $\approx$ an equivalence relation on X. For any element a of X, let $X_a = \{x \in X : x \approx a\}$. Show that:

(i) $X_a \neq \emptyset$ for all a,

(ii) if $X_a \cap X_b \neq \emptyset$, then $X_a = X_b$,

and (iii) $X = \bigcup_a X_a$.

The set X_a is called the **equivalence class** of a. A collection of subsets $\{X_a\}$ of a set X having properties (i), (ii), and (iii) is called a **partition** of X.

(f) if X is a given set and $\{U_\alpha\}$ is a partition of X, show that there is an equivalence relation $\approx$ on X so that $\{U_\alpha\}$ is the collection of equivalence classes of $\approx$.

(g) Show that the conditions defining an equivalence relation are independent by giving three examples of relations having two of the properties but not the third.

2.2 INFINITE SETS

Though we have a definition of what it means for a set to be infinite, we don't know whether there are any infinite sets!

THEOREM 2.4: *The set of all natural numbers,* **N**, *is infinite.*

PROOF: We show that **N** can't be put in one-to-one correspondence with any initial segment of **N**. Suppose $I = \{1, 2, \ldots, n\}$ is an initial segment and that $c : I \to \mathbf{N}$ is a function (it doesn't matter now whether c is one-to-one or not). The collection of outputs of c is finite since it is in one-to-one correspondence with I. Let $s = c(1) + c(2) + \cdots + c(n) + 1$. Then s is a natural number that is larger than any of the outputs of c. Thus $s \in \mathbf{N}$ is not an output of c, and c is not onto. No function from any initial segment to **N** can be onto, and consequently **N** is not finite. ∎

In the context of cardinality, the words "larger" and "smaller" take on new meanings. Ordinarily, we would say that the set of natural numbers is larger than the set of even natural numbers because the former contains all the elements of the latter and more. But in the sense of cardinality, these two sets are the same size. The footnote to Exercise 2.1.3 suggests a way of sorting this out. We will say that set A has **smaller cardinality** than set B if no function from A to B is onto, and that A has **larger cardinality** than B if no function from A to B is one-to-one.[3]

Here is a more striking example. The set of rational numbers, **Q**, seems to be very much larger than the set of natural numbers. We will show that these two sets have the same cardinality! We begin by showing that the set of *positive* rational numbers has the same cardinality as **N**.

---

[3] These definitions leave us with still another question: If A has smaller cardinality than B, does B necessarily have larger cardinality than A? The answer is Yes, but this is remarkably difficult to show. If we take our definition to be "B has larger cardinality than A if A has smaller cardinality than B," we are left with an equally difficult question about functions.

Here are the positive rational numbers:

1/1	1/2	1/3	1/4	...
2/1	2/2	2/3	2/4	...
3/1	3/2			
⋮	⋮			

We will match the natural numbers with the entries in this table in a way that is one-to-one and onto. Starting at the upper left, associate 1 with 1/1. Move down, to 2/1. Check whether this rational number has already been assigned some natural number. It hasn't, so we assign to it the next natural number, 2. If it has already been assigned a natural number, just skip over it. Move up and to the right, to 1/2, and do the same thing. Proceed through the table in the pattern shown in the next diagram. This describes the desired one-to-one correspondence (we don't always need a formula to have a function) and shows that the set of positive rational numbers has the same cardinality as the set of natural numbers. You will finish the proof in Exercise 2.2.1. (Note that 2/2 has already been assigned a natural number when we get to it.)

1 1/1	3 1/2	5 1/3	1/4	...
2 2/1	~~2/2~~	2/3	2/4	...
4 3/1	3/2			
⋮	⋮			

We are generally interested in sets with only three types of cardinalities: Natural numbers (the cardinalities of finite sets), the cardinality of **N** (which is denoted $\aleph_0$—aleph zero or aleph naught[4]), and those that are bigger. A set with cardinality $\aleph_0$ is said to be **denumerable**. A set that is finite or denumerable is said to be **countable**. If a set is not countable, it is **uncountable**.

Upon careful examination, we see that the proof of the countability of the rational numbers establishes more than we claimed for it. It really shows that *the union of any denumerable collection of denumerable sets is denumerable*. There are several ways to state a result like this, depending on precisely how many sets there are and how many elements each of them

[4] $\aleph$ is the first letter of the Hebrew alphabet. $\aleph_0$ is the "first infinity."

has. They can be summed up in the following very useful theorem, whose proof is Exercise 2.2.2.

THEOREM 2.5: *The union of a countable collection of countable sets is countable.* ∎

EXERCISES 2.2

1. (a) Complete the proof that $\mathbf{Q}$ has cardinality $\aleph_0$.

   (b) While proving that the positive rational numbers are denumerable we might observe that, if we knew that the Cantor-Bernstein-Dedekind theorem was true (see Exercise 2.1.3), we wouldn't have to fuss over whether or not we had already counted each of the rational numbers when we get to it in the array. Explain.

2. (a) State the results that are included in Theorem 2.5 (there are four that are easy to state, and others that are more complicated).

   (b) Prove Theorem 2.5.

3. Show that a set is countable if and only if it can be put in one-to-one correspondence with a subset of $\mathbf{N}$.

4. (a) If d_1 is a digit (one of the symbols 0, 1, ..., 9), how many decimals are there of the form $0.d_1000\ldots$? (Say *exactly.*)

   (b) How many decimals are there of the form $0.d_1d_2000\ldots$? (Say *exactly.*)

   (c) How many decimals are there of the form $0.d_1d_2\ldots d_n000\ldots$ for a given n?

   (d) Show that there are countably many terminating decimals.

5. If a number is a solution to a polynomial equation with coefficients that are integers (for example, $3x^2-7x+19=0$) it is called **algebraic**. For instance, $\sqrt{2}$ is algebraic since it is a solution to $x^2-2=0$.

   (a) Show that all rational numbers are algebraic.

   (b) For any n, show that the collection of polynomial equations with integer coefficients and degree less than n is countable.

   (c) Show that the set of algebraic numbers is countable.

6. The expression $2+3=5$ may be interpreted "The union of disjoint sets with two and three elements is a set with five elements," while $2\times3=6$ says "The Cartesian product of a set with 2 elements and a set with 3 elements is a set with 6 elements." We may (loosely) interpret the

expression $\aleph_0 + 1 = \aleph_0$ to mean "The union of a denumerable set with a one-element set is denumerable." Interpret each of the following in words, and prove them:

(a) $\aleph_0 + \aleph_0 = \aleph_0$.

(b) $\aleph_0 + n = \aleph_0$ for any $n \in \mathbf{N}$.

(c) $\aleph_0 \times \aleph_0 = \aleph_0$.

(d) Why does the first pair of sets mentioned in this problem have to be disjoint, while the second pair does not?

7. (a) The set of ordered pairs of elements of a set S is called (after René Descartes) the **Cartesian product of S with itself** and is denoted $S \times S$. Use a technique similar to that used to prove the countability of the rational numbers to show that $\mathbf{N} \times \mathbf{N}$ has the same cardinality as $\mathbf{N}$.

(b) Show that the function given by

$$F(i,j) = \binom{i+j-1}{2} + j$$

maps $\mathbf{N} \times \mathbf{N}$ one-to-one and onto $\mathbf{N}$. The expression $\binom{n}{k}$ is referred to as a **binomial coefficient**, often pronounced "n choose k." (Note that it is *not* $\frac{n}{k}$.) It is the coefficient of x^k in the binomial expansion $(1+x)^n$. It is taken to be 0 if $n < k$ and is given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

otherwise.

(c) Examine the pairs of numbers that get mapped to 1, 2, ... by the function in (b). Is there a connection between your proof of (a) and the function in (b)?

2.3 UNCOUNTABLE SETS

We find ourselves in a familiar position. We have defined what it means for a set to be "uncountable," but we don't know whether any uncountable sets exist. Now we will show that there is one. Consider the set of nonterminating decimals with all digits to the left of the decimal point zero. (It wouldn't hurt to think of these as the real numbers between 0 and 1, but we have yet to explore the relationship between decimals and the real numbers.) A function whose range is this set and whose domain is the natural numbers may be thought of as an infinitely long list:

$$\begin{array}{ccc} 1 & \to & 0.63548\ldots \\ 2 & \to & 0.30126\ldots \\ 3 & \to & 0.11670\ldots \\ 4 & \to & \ldots. \end{array}$$

We will construct a decimal that is not on this list (that is, one that is not in the range of this function). If the *first* place of the *first* entry on the list is 6 (it is), make the first place of the new decimal 7. If the first place of the first entry is not 6, make the first place of the new decimal 6 (any two new digits will do, but we should not use 0 for one of them). Now look at the *second* decimal place of the *second* entry on the list and select the second place of the new decimal in the same way. Move on to the third place of the third entry, and so on. Our new decimal begins 0.767.... This is not one of first three entries on the list, and in fact the whole decimal constructed in this way is *not on the list at all* since it differs from the nth entry on the list at least in the nth decimal place. We have shown that *no function from* $\mathbf{N}$ *to the set of such decimals is onto*, and therefore that this set is *uncountable*. (Notice that it doesn't matter whether the function that describes the list is one-to-one.)

This process is called **Cantor diagonalization**, after Georg Cantor, who first studied these ideas extensively in the late nineteenth century. His theories were vigorously denounced by prominent authorities of the time, but quickly won out as their essential correctness was understood by more people. The birth of this subject is well worth investigating if you believe that scientists are never motivated by pettiness, or that there is no real connection between mathematics and truth.

EXERCISES 2.3

1. Explain the remark "any two new digits will do, but we should not use 0 for one of them" in the description of Cantor diagonalization.

2. Why would anyone object to the ideas of cardinality?

3. Here is a recreational application of the idea of uncountability.

   (a) Show that there are uncountably many theorems in mathematics.

   (b) "Mathematical English" can be written with a countable collection of symbols. Show that for a given natural number n, there are only countably many proofs that can be written in fewer than n symbols.

   (c) Show that for a given natural number n, there is a theorem whose proof requires more than n symbols to write.

4. The **power set** of a set S, denoted $P(S)$, is the set of all subsets of S.

   (a) List the entire power set of $S = \{a, b, c, d\}$.

   (b) Define a function from this set S to $P(S)$ [simply pick sets to be $f(a)$, $f(b)$, $f(c)$, and $f(d)$]. Note that some elements of S are also elements of their image under your function, while others are not (either of these groupings might be empty). Make a list of the elements that are *not* elements of their images under your function (this list might also be empty). Note that the set you have just written down is not the image of any element under your function. Define another function and repeat this. Observe that the set you produce in the end never is an output of your function.

   (c) Here we show that if S is any set at all, the cardinality of $P(S)$ is larger than the cardinality of S. Suppose $f : S \to P(S)$. Show that $\{x : x \notin f(x)\}$ is not in the image of f. Explain how this proves that the cardinality of a power set is larger than the cardinality of the original set.

5. (a) Show that there are infinitely many different infinite cardinals.

   (b) Are there uncountably many infinite cardinals?

6. (a) Show that any set having an infinite subset is infinite.

   (b) Show that any set having an uncountable subset is uncountable.

   (c) If $A \subseteq B$, show that the cardinality of B is not smaller than the cardinality of A.

7. (a) Show that the natural numbers can be put in one-to-one correspondence with a proper subset of the natural numbers.

   (b) Give a plausible justification of the statement "Every infinite set has a denumerable subset."[5]

   (c) Show that every infinite set can be put in one-to-one correspondence with a proper subset of itself (this is an alternative definition of "infinite"). Think about the real number line as an example. Compare this with the result of Exercises 2.1.4.e.

   (d) Show that forming the union of an infinite set with a finite set does not increase the first set's cardinality.

   (e) Show that forming the union of an infinite set with a countable set does not increase the first set's cardinality.

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[5] The *proof* of this statement is beyond the scope of this book. Examine your answer carefully and pick out the assumptions you must make about set theory to make it work.

8. Assuming for the moment that the set of real numbers is uncountable, show that there must be nonalgebraic numbers (see Exercise 2.2.5). Numbers that are not algebraic are **transcendental**. The familiar numbers e and π are transcendental, but the latter fact, especially, is difficult to prove. (That π is transcendental was not proven until 1882.)

9. (a) Show that the set of subsets of a denumerable set is uncountable.

   (b) Show that the set of *finite* subsets of a denumerable set is denumerable.

   (c) Use (b) to show again that the set of terminating decimals is countable. (There is more work here than there might seem—be careful.)

10. Carefully consider the difficulties that arise in Cantor diagonalization if one allows terminating decimals. Can the process be "patched up" to allow terminating decimals but avoid these problems?

Chapter 3

Algebra of the Real Numbers

3.1 THE RULES OF ARITHMETIC

Just what is a real number? This is a deceptively subtle issue. Any guess we might make now would likely be, at best, incomplete. Failing to find an answer to this question, we might ask a simpler one: What can we *do* with real numbers? We know plenty about this. We can add and subtract, multiply and divide. The associative, commutative, and distributive laws, and such, occupied our attention for months when we were younger. Whatever the real numbers actually are, we know there are operations on them known as addition and multiplication that obey rules like these:

(0) If we add or multiply two real numbers, we get a real number.

(1) **Addition is associative:** If x, y, and z are any three real numbers, then $(x+y)+z = x+(y+z)$.

(2) **Addition is commutative**: If x and y are any two real numbers, then $x+y=y+x$.

(3) There is a special real number, 0 (the **additive identity**), having the property that $0+x=x$ for any real number x.

(4) For each real number, x, there is a real number, denoted $-x$, with $x+(-x)=0$ ($-x$ is the **additive inverse** of x).

(5) **Multiplication is associative.**

(6) **Multiplication is commutative.**

(7) There is a special real number, 1 (the **multiplicative identity**), having the property that $1 \times x = x$ for every real number x.

(8) For each real number except 0, there is another real number, denoted $1/x$ or x^{-1}, with $x \times x^{-1} = 1$ (x^{-1} is the **multiplicative inverse** of x).

(9) **Multiplication distributes over addition**: If x, y, and z are any numbers, then $x \times (y+z) = (x \times y)+(x \times z)$.

There are other rules of arithmetic, of course. For instance, if a and b are real numbers and a is not 0, there is a real number x so that $a \times x = b$. This rule (which gives us the important ability to solve equations) is not on the list because we can prove it from things that are already there. We want the list to be as short as possible.

We use some rules so much that we might not even realize they are rules. For instance, we always use 0 as the additive identity, and it might never occur to us that another number might do that job just as well. Could this be? Suppose z works as an additive identity. Then $0 + z = 0$ (because z works this way). But $0 + z = z$ (because 0 works this way). Now $0 + z$ can be only one thing, and so it must be that $z = 0$. We have shown that *There is only one additive identity*. Arguments like this are part of "algebra" (a large subject of which factoring polynomials and the like are small parts).

3.2 FIELDS

Unfortunately, the rules of arithmetic have nothing to say about our common notions of what real numbers *are*. They make no mention of decimal expansions or number lines; they offer no hint of *how* to add or multiply. We can put this lack of specificity to work for us. Notice, for instance, that all of the rules remain valid if we replace the word "real" with the word "rational" wherever it appears. Perhaps there are still other structures that obey these rules. What about the integers? We can add them and multiply them, but integers generally do not have multiplicative inverses. Some familiar structures obey these rules, some don't. We begin the process of generalization by giving a name to "anything that obeys the rules."

DEFINITION 3.1: A set **F**, with operations + and ×, obeying rules (0) through (9) above (with "real number" replaced by "element of **F**" everywhere it occurs) **is called a field.**

Most of our early mathematical education consisted of discussions of the field structure of the rational numbers and real numbers. Remember that one of our goals in this book is to discover ways in which these two particular fields are different, and algebra can often be used to make such distinctions. The set of rational numbers, **Q, is a field.** The set of integers, **Z, is not**. We gather from this that **Q is different from Z** (that some integers *look* different from some rational numbers is not sufficient evidence; see Exercise 3.2.5). But algebra alone doesn't allow us to distinguish the set of rational numbers from the set of real numbers. They

are simply both fields.

Are there any other fields? Consider the two-element set $\{0, 1\}$ and define two operations as follows (we will circle the symbols to remind us that they are not ordinary addition and multiplication, even though the things being "added" and "multiplied" look like numbers):

$$\begin{array}{ll} 0 \oplus 0 = 0 & 0 \otimes 0 = 0 \\ 0 \oplus 1 = 1 \oplus 0 = 1 & 0 \otimes 1 = 1 \otimes 0 = 0 \\ 1 \oplus 1 = 0 & 1 \otimes 1 = 1 \end{array}$$

It is easy to check that this structure obeys all the rules that define a field. **This field is called $\mathbf{Z}_2$.**

Why would anyone do this sort of thing? Look at our proof that the real numbers have only one additive identity. It involved only the rules for a field, not any specific knowledge about real numbers. It works *just as it is* for any field, and so *there is only one additive identity in any field.* No matter how often we encounter fields, we need never prove this again. The generality of algebraic proofs gives them much power. We now know of three fields: **Q**, **R**, and $\mathbf{Z}_2$. Here are two more (you will check the details in Exercise 3.2.12).

EXAMPLES 3.2: 1. The **formal rational functions** are the usual quotients of polynomials, but we don't concern ourselves with annoying details like their domains. For instance, $\frac{7x^4+2x^3-4x^2+14x-9}{x^3-6x^2+3x+11}$ is a formal rational function *because of what it looks like* (that is, because of its *form*[1]). We already know how to add and multiply rational functions, and it is easy to check that this is a field.

2. The set **C** of **complex numbers** is defined by endowing a special symbol, usually i (electrical engineers use j, because i means something else to them), with the property $i^2 = -1$ (we shall see later that no real number has this property). Complex numbers are written $a + bi$, where a and b are real. Operations are done as if i is a variable, with i^2 replaced by -1 whenever it occurs, for instance:

$$\begin{aligned} & (4+i)\times(2+3i) \\ = \; & (4)(2)+(2)(1)i+(4)(3)i+3i^2 \\ = \; & 8+14i-3 \\ = \; & 5+14i. \end{aligned}$$

[1] This is an example of the mathematical usage of the word "formal." It means that we should ignore details that we would otherwise consider important. In other contexts such a discussion would be called *informal*!

While it is fairly easy to check that these structures are fields, it is not so easy to see that they are different in any significant way from the real or rational numbers. We will be able to make these distinctions later by examining another kind of structure. We should keep in mind that the real numbers and the rational numbers play important roles in many areas of mathematics. We will refer freely to their algebraic properties, but in this book we will never again pause and say "Let's see if that proof works for any field" like we have here. This kind of thinking, though, is a basic *modus operandi* of the algebraist:

(1) While studying some familiar object, pause to write down rules that govern what you have learned.

(2) Examine your list to see if there is any redundancy. Is there anything that can be proved from other things on the list? Toss anything that can off the list.

(3) Give a name to "anything that obeys the rules." At this point the rules of one subject become definitions for another.

(4) Study the things you've just named. The first thing you will want to do is to look for "something that obeys the rules" that is different from the objects you were thinking about when you made the list.

There is another very important process of algebra that we haven't seen. It goes like this: We know something about fields now. How important are, say, multiplicative inverses, anyway? We can add and multiply integers, and everything seems to work just fine. What happens if we remove rule (8) from the definition of a field? "Things that work this way" are not necessarily fields. Give them a new name and start up again!

EXERCISES 3.2

1. We know that the integers and the natural numbers are not fields, but which, if any, of the rules for a field *do* each of these sets satisfy?

2. If x is an element of a field such that $x^2 = x$, show that either $x = 0$ or $x = 1$.

3. If a and b are elements of a field $\mathbf{F}$ and $a \neq 0$, show that there is an $x \in \mathbf{F}$ so that $a \times x = b$.

4. If a and b are elements of a field, show that $-(a+b) = -a - b$.

5. (a) Verify that $\mathbf{Z}_2$, as described in the chapter, is a field.
   (b) Let $\mathbf{F} = \{a, b\}$, and define operations on $\mathbf{F}$ by:

$$a \oplus a = a \qquad a \otimes a = a$$
$$a \oplus b = b \oplus a = b \qquad a \otimes b = b \otimes a = a$$
$$b \oplus b = a \qquad b \otimes b = b$$

Show that $\mathbf{F}$ is a field.

(c) Explain how this field is similar to $\mathbf{Z}_2$.

(d) Show that, other than changing their names, this is the *only* way to make a field of a set with two elements.

6. $\mathbf{Z}_5$ is the set $\{0, 1, 2, 3, 4\}$ with arithmetic done modulo 5, that is, do the usual operations and then subtract 5 repeatedly until the result is an element of the set. We can do arithmetic modulo any natural number greater than 1, so $3 + 4 = 2 \pmod 5$, $5 \times 6 = 6 \pmod 8$, and $9 \times 8 = 0 \pmod{12}$, for example.

   (a) Show that $\mathbf{Z}_5$ (or $\mathbf{Z}_3$ or $\mathbf{Z}_7$) is a field.

   (b) Show that $\mathbf{Z}_4$ (or $\mathbf{Z}_6$ or $\mathbf{Z}_{14}$) is *not* a field.

   (c) For which values of n is $\mathbf{Z}_n$ a field? State and prove a theorem.

7. Show that each element of a field has only one additive inverse and that each nonzero element has only one multiplicative inverse.

8. Show that $0 \times x = 0$ for any x in any field.

9. (a) Consider the set $\mathbf{D} = \{d\}$, with addition and multiplication given by $d + d = d$ and $d \times d = d$ (this is the only way the operations could be defined on such a set). Show that $\mathbf{D}$ is a field. What is the additive identity in this field? What is the multiplicative identity?

   (b) In the field $\mathbf{D}$ given in (a), we have $0 = 1$ (the additive identity equals the multiplicative identity), which is inconvenient. Show that $0 \neq 1$ in any field with more than one element. (A field with only one element is not very interesting, and most algebraists include in their definition a stipulation that a field must have at least two elements.)

   (c) In the field $\mathbf{D}$ given in (a), multiplication and addition are the same (the product of any two elements is the same as their sum). This must be true, of course, in a field with only one element. Are there any other fields in which this is the case?

10. Show that $(-1) \times x = -x$ in any field. (Note that $(-1) \times x$ is the product of the additive inverse of 1 with x, while $-x$ is the additive inverse of x.)

11. If x is an element of a field, show that $(-x)^2 = x^2$ (where $x^2 = x \times x$).

12. (a) Show that $\mathbf{C}$ is a field. In particular, find a formula for $(a+bi)^{-1}$.

    (b) Show that the set of formal rational functions is a field.

13. The structure described in the last paragraph of the chapter is called a **commutative ring with unity**. "Commutative" refers to the multiplication operation, and "unity" is the multiplicative identity.

    (a) Show that the integers and the set of polynomials are commutative rings with unity.

    (b) Guess the definition of "ring." (What happens when you leave out the adjectives?)

    (c) (If you've taken linear algebra ...) Show that the set of 2×2 matrices is a ring with unity (but is not commutative).

    (d) One of the axioms deleted from the definition of "field" to obtain the definition of "ring" is the existence of multiplicative inverses. Is the set of *invertible* 2×2 matrices a field?

    (e) Give an example of a ring *without* unity.

14. Here is more Linear Algebra: Let $\mathbf{M}_2$ consist of all 2×2 matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, where a and b are real numbers.

    (a) Show that $\mathbf{M}_2$ is a field under ordinary addition and multiplication of matrices.

    (b) Find the multiplicative identity in $\mathbf{M}_2$.

    (c) Find the square of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

    (d) Discuss the relationship between $\mathbf{M}_2$ and $\mathbf{C}$.

15. It is an underlying assumption of this text that we understand how the rational numbers work. Just to be sure ...

    (a) Show that the sum and product of two rational numbers is a rational number.

    (b) Show that the sum of a rational number and an irrational number is an irrational number.

    (c) Is (b) also true for products?

    (d) Is (a) true for pairs of irrational numbers?

    (e) Show in detail that $\mathbf{Q}$ is a field.

    (f) Is the set of irrational numbers a field?

16. (a) Let $\mathbf{F}$ be the set of all real numbers of the form $a + b\sqrt{2}$, where a and b are rational numbers. Show that $\mathbf{F}$ is a field.

(b) If $k > 0$ is a rational number that doesn't have a rational square root, show that the set $\{a + b\sqrt{k} : a, b \in \mathbf{Q}\}$ is a field.

(c) If the number k in (b) *does* have a rational square root, show that the set constructed is just $\mathbf{Q}$.

(d) Suppose k is a rational number that doesn't have a rational square root (this time, k might be negative). We endow the symbol $\diamond$ with the property that $\diamond^2 = k$. Show that the collection of symbols $a + b\diamond$ is a field (note that they may not be real numbers). Here multiplication and addition are carried out as if $\diamond$ were a variable, with $\diamond^2$ replaced by k whenever it appears. For instance, we would have:

$$(1 + \diamond)(2 + \diamond) = 2 + 3\diamond + k.$$

(e) Suppose $\mathbf{F}$ is any field and $k \in \mathbf{F}$ is such that there is no element of $\mathbf{F}$ whose square is k. Repeat part (d) of this exercise in this setting.

17. According to the previous exercise, the collection of symbols $a+b\sqrt{-5}$, where a and b are rational numbers, is a field.

(a) Define an "absolute value" on this field by saying $\|a + b\sqrt{-5}\| = \sqrt{a^2 + 5b^2}$. Show that this function resembles the absolute value in the sense that, if x and y are in this field, then $\|x \times y\| = \|x\| \times \|y\|$.

(b) We may define "integers" in this field to be those elements where a and b are both integers. Show that, if x and y are "integers" and x divides evenly into y, then $\|x\| \le \|y\|$.

(c) Show that $1 + \sqrt{-5}$ and $1 - \sqrt{-5}$ are "prime" in this field in the sense that the only numbers that divide evenly into either of these are 1 and the element itself. Show that 2 and 3 are also prime in this field.

(d) Show that $6 = 2 \times 3 = (1 + \sqrt{-5}) \times (1 - \sqrt{-5})$ in this field. (So it is possible to factor 6 into primes in more than one way.)

Chapter 4

Ordering, Intervals, and Neighborhoods

4.1 ORDERINGS

When we first learned about number systems, "less" and "greater" meant "to the left" and "to the right" on the big number line above the blackboard. Together with pictorial interpretations of addition and multiplication, this view served its purpose quite well, and we learned a lot from it. Our understanding of algebra has matured, though, and we need a more precise idea of the ordering of the real numbers to go along with it.

We have a lot of choices when we set out to impose an ordering on a set. Even the simplest of orderings can be described in more than one way, and we are free to pick a description that suits our purposes. But no matter how we describe the ordering of a set, our goal is to be able to pick any ordered pair of elements of the set, (a, b), and say whether the statement $a < b$ is true or false. The **ordering** itself is considered to be the set of pairs for which $a < b$ is true. (The ordering technically focuses on the "$<$" symbol, but of course we also should know what "$>$" means.)

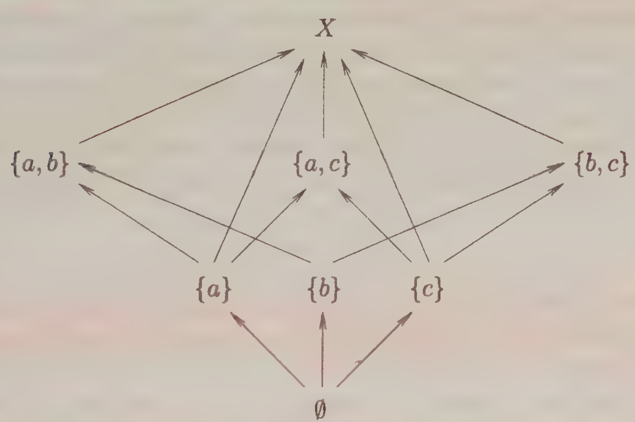
EXAMPLES 4.1: 1. We could order words by simply counting their letters. Under such an ordering, we have, for instance, BIG $<$ SMALL and FOUR $<$ THREE. Many words that are not equal are also neither greater than nor less than each other under this ordering.

2. Words are usually ordered by **lexicographic** (or **dictionary**) ordering. To decide which of two words is "smaller," we compare their first letters alphabetically. If the first letters are the same, we compare the second letters, and so on. To compare words of different lengths, we decree that the shorter word ends with a blank, which we take to be at the beginning of the alphabet. For instance, HEDRIN $<$ HEPBURN and CAT $<$ CATACLYSM. Here is a portion of this ordering from one dictionary:

moonstruck < moonwalk < moonwort < moony < moor < moorage.

Elements of an ordered set are said to be **comparable** if one of the statements $a < b$, $a > b$, or $a = b$ is true. Observe that, unlike the previous example, every word is comparable to every other word in the lexicographic ordering (of course, there might be disagreement about which collections of letters are words—"moonwalk" certainly doesn't appear in all dictionaries). Notice also that in this ordering (in this dictionary), there is no word between "moonwort" and "moony." We will see that these are useful properties for an ordering to have.

3. Let $X = \{a, b, c\}$. We may define an ordering on $P(X)$ (the power set of X) by saying $S < T$ if $S \subset T$ and $S \neq T$. Then, for instance, $\{a\} < \{a, b\}$. The $<$ relation can be described in a diagram:



Here, $A < B$ if it is possible to go from A to B by moving upward along arrows. Note that there are no horizontal arrows, because, for instance, neither $\{a\} < \{b\}$, $\{b\} < \{a\}$, nor $\{a\} = \{b\}$ is true. In this ordering, $\{a\}$ and $\{b\}$ are not comparable.

4. The decimals we introduced when discussing uncountable sets can also be ordered lexicographically. However, if we insist on thinking of them as the real numbers between 0 and 1, we get some surprising results. For instance, $0.37455 < 0.702$, as we would expect. However, we have $0.4999\ldots < 0.5$ under the lexicographic ordering, where we would expect these two decimals to be equal. In the lexicographic ordering of these decimals, as with words in the dictionary, there are pairs of different elements with no intervening element. This also does not happen in the usual ordering on the real numbers. Changing the ordering of a set can

affect its structure dramatically.

All the orderings we will study have the property mentioned in Example 4.1.2: Every element is comparable to every other element. The name we give this property is suggested by the "line" of words in the example. It is with this image in mind that we sometimes refer to the set of real numbers as the "real number line," and to real numbers as "points."

DEFINITION 4.1: An ordering on a set is **linear** if, for any pair of elements of the set (a, b) one and only one of the following holds:

$$\begin{aligned} &\text{(i)}\ a < b; \\ &\text{(ii)}\ a = b; \\ \text{or}\ &\text{(iii)}\ b < a. \end{aligned}$$

This is called the **trichotomy**, a word we learned as school children mainly because it sounds so fancy. While the orderings described in the examples above might be useful for some purposes, there does not seem to be any connection between what the elements of the sets *are* and the ordering imposed on them. In the next section we will see an example where the ordering of a set and the meaning of its elements come together.

EXERCISES 4.1

1. If S is a subset of an ordered set, a **least element** of S is an element x, if there is one, such that (i) $x \in S$ and (ii) if $y \in S$ and y is comparable to x, then $x \leq y$.([1])

   (a) Show that a subset of a linearly ordered set can have at most one least element.

   (b) Show that in an ordering that is not linear a set can more than one least element.

   (c) Show that a subset of a linearly ordered set might not have a least element.

2. Construct examples of ordered sets for which various combinations of parts of the trichotomy fail.

3. Are the decimals we discussed, with the lexicographic ordering, linearly ordered? What if we consider the decimals to be real numbers? (Think about 0.4999... and 0.5.)

---

[1] The symbol $\leq$ means just what we think it does: $a \leq b$ if $a < b$ or $a = b$.

4. The following "proof" contains at least two serious errors. Find them and say why they are errors; then give an example to show that the result is false. (Note that the fact that the proof is incorrect is not enough to guarantee that the result is false. On the other hand, the fact that the result is false means that the proof must be incorrect, though knowing this may not help us find the errors.)

   "THEOREM": Given any infinite subset of the real line, S, there are infinitely many pairs of numbers (a, b) such that $a < b$ and $a, b \in S$ but $\frac{a+b}{2} \notin S$

   "PROOF": Let $S = \{s_1, s_2, \ldots\}$. Since $s_1 < s_2 < \ldots$, each of the pairs of numbers (s_n, s_{n+1}) satisfies the conditions of the theorem.

5. (a) How many ways are there to impose an ordering on a set with two elements? Three elements? N elements?

   (b) How many of these orderings are linear?

6. If we agree that the denominators used in representing rational numbers should always be positive, their usual ordering is given by

$$\frac{p}{q} < \frac{r}{s} \Leftrightarrow ps < qr.$$

   Show that this is a linear ordering.

7. Let $\{S_\alpha : \alpha \in \mathcal{A}\}$ be a collection of sets and suppose that $\mathcal{A}$ is linearly ordered. Define the limit supremum and limit infimum of the collection $\{S_\alpha\}$ as in Exercise 1.15.9 and generalize the results there.

8. If X is any set, the power set of X can be ordered as in Example 4.1.3, by saying that $S < T$ if $S \subset T$ and $S \neq T$.

   (a) Is this *ever* a linear ordering?

   (b) Show that an ordering constructed in this way has the following property: If $S, T \in P(X)$, then $\exists U \in P(X) \ni (S < U$ and $T < U)$. This says: "S and T may not be comparable, but there is something comparable to, and bigger than, both of them." A set with an ordering having this property is called a **directed set**.

   (c) Suppose we order $P(X)$ by saying $S < T$ if $S \supset T$ and $S \neq T$ (notice the change!). Show that X is a directed set with this ordering.

4.2 THE ORDERING OF THE NATURAL NUMBERS

Even something as familiar as the usual ordering on the natural numbers may be described in more than one way. The first of these definitions is the more traditional, while the second is more closely related to our work in Chapter 2. You will show in Exercise 4.5.2 that the definitions are equivalent.

DEFINITION 4.2: If m and n are natural numbers, we say $\boldsymbol{m < n}$ if either

(a) n is among the natural numbers: $m + 1, m + 1 + 1, m + 1 + 1 + 1, \ldots$

or

(b) no function whose domain is a set with m elements and whose range is a set with n elements is onto.

4.3 WELL-ORDERING AND INDUCTION

Definition 4.2 allows us to indicate (if not list) all the pairs in the ordering of the natural numbers. We will not spend time discussing the ordering of the natural numbers per se. Our interest is in a special property it has.

DEFINITION 4.3: A linearly ordered set is said to be **well-ordered** if every nonempty subset of it has a least element.[2]

The rational numbers are not well-ordered. For instance, $\{p \in \mathbf{Q} : p \geq 0\}$ has a least element, but $\{p \in \mathbf{Q} : p > 0\}$ does not. *That the natural numbers are well-ordered is an axiom.* If a set of natural numbers is given to us in a list, it is easy to pick out its least element. The least element of $\{12, 6, 3, 173\}$ is 3. On the other hand, if a set is described in some way, picking out its least element may not be so simple. What is the least element of the set of natural numbers that can be written as a sum of three primes but can't be written as a sum of two primes? Are there any? This may take some thought. We will come to see, however, that well-ordering helps us not so much by picking out the least element of a set as by guaranteeing that there is one. If we can find *one* natural number fitting some description, we know there is a *least* number fitting that description.

As an illustration of the importance of well-ordering, we will prove the following theorem, which gives us an important technique of proof called **induction**.

[2] The least element of a set was defined in Exercise 4.1.1.

THOEREM 4.4: *Suppose $S \subseteq \mathbf{N}$ is such that*

$$\begin{aligned} &\text{(i)} \quad 1 \in S \\ \text{and} \quad &\text{(ii)} \quad k \in S \Rightarrow k+1 \in S \text{ whenever } k \geq 1. \end{aligned}$$

Then $S = \mathbf{N}$.

PROOF: Suppose $S \neq \mathbf{N}$ and let $T = \mathbf{N} \setminus S$. Since $S \neq \mathbf{N}$, we have $T \neq \emptyset$. By the well-ordering property, T has a least element; call it t. Now $t \neq 1$ because $1 \in S$. Let $s = t - 1$. Since $t > 1$, s is a natural number. Since t is the *least* element of T and $s < t$, s can't be in T, and so $s \in S$. But then $t = s + 1 \in S$, a contradiction. ∎

The well-ordering property and the validity of induction are equivalent (you will verify this in Exercise 4.5.7). Either may be taken as an axiom of the natural numbers, with the other being a theorem.

EXAMPLES 4.3: 1. Induction is often used to prove arithmetic formulas. Let us show that $1 + 2 + \cdots + n = n(n+1)/2$. Let S be the set of natural numbers for which this is true. Now $1 \in S$ since $1 = 1(1+1)/2$. Suppose $k \in S$. Then

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

To show that $k + 1 \in S$, we must show that

$$\begin{aligned} & 1 + 2 + \cdots + (k+1) \\ = \; & \frac{(k+1)(k+2)}{2} \end{aligned}$$

But note that

$$\begin{aligned} & 1 + 2 + \cdots + (k+1) \\ = \; & 1 + 2 + \cdots + k + (k+1) \\ = \; & \frac{k(k+1)}{2} + (k+1) \\ = \; & \frac{(k+1)(k+2)}{2}, \end{aligned}$$

which is just what we want. ∎

2. Induction can also be used to prove general statements about finite sets. We will show that *every finite set has a largest element*. This result pops up repeatedly in our work (we could have used it to prove Theorem 2.4). Let S be the set of natural numbers for which "Any set

with exactly n elements has a largest element." Clearly $1 \in S$. Suppose $k \in S$ (so any set with exactly k elements has a largest element) and let T be a set with exactly $k+1$ elements. Let $t \in T$. Then $T \backslash \{t\}$ has exactly k elements, and so it has a largest element, say b. If $t > b$, then t is the largest element of T. If $t < b$, then b is the largest element of T. In either case, T has a largest element since both t and b are in T.

3. Induction is a powerful tool that must be used with care. Failure to exercise proper caution can lead to some curious results. For instance: Let S be the set of natural numbers for which the statement "In any set with exactly n elements, all the elements are the same" is true. We will show that $S = \mathbf{N}$. (Roughly translated, this means "*Among all the objects in the universe that can possibly be elements of any finite set, there is only one thing.*") This appears to be false, but .... Clearly $1 \in S$ (in any set with exactly 1 element, all the elements are certainly the same). Suppose that $k \in S$ (that is, in any set with exactly k elements, all the elements are the same) and let T be a set with exactly $k+1$ elements, say $T = \{t_1, t_2, \ldots, t_{k+1}\}$. Let $T_1 = T \backslash \{t_1\}$ and $T_2 = T \backslash \{t_2\}$. Now T_1 and T_2 are both sets with exactly k elements, and therefore all the elements of T_1 are the same and all the elements of T_2 are the same. But $t_{k+1} \in T_1 \cap T_2$, and so all the elements of T_1 are the same as t_{k+1} and all the elements of T_2 are the same as t_{k+1}. Therefore all the elements of T are the same since they are all the same as t_{k+1}. Something fishy is going on here (unless you believe the result!). You will explain what it is in Exercise 4.5.8.

In both of the real examples of induction, the set in question is taken to be the set of natural numbers for which some open statement is true. We use this observation to state induction in a form more useful for our purposes:

Theorem 4.5: *Suppose $P(n)$ is an open statement, where n can be any natural number. If*

$$\begin{aligned} &\text{(i)} \quad P(1) \text{ is true} \\ \text{and} \quad &\text{(ii)} \quad P(k) \Rightarrow P(k+1) \text{ whenever } k \geq 1, \end{aligned}$$

then $P(n)$ is true for all $n \in \mathbf{N}$.

PROOF: All we need to do is generalize the arguments in the examples. Let $S = \{n \in \mathbf{N} : P(n) \text{ is true}\}$. Suppose that $S \neq \mathbf{N}$ (that is, that there is a natural number n for which $P(n)$ is false). Then $\mathbf{N} \backslash S$ has a least element, say n_0. Since $P(1)$ is true, we know $n_0 \neq 1$. Then $P(n_0 - 1)$ is true [since $n_0 - 1 < n_0$, and n_0 is the *least* natural number for which $P(n)$ is false]. By (ii), it follows that $P(n_0) = P(n_0 - 1 + 1)$ is also true,

a contradiction. ∎

Notice that statement (ii) of Theorem 4.5 is an implication. Its hypothesis is called the **induction hypothesis**. The moment in a proof at which this implication is established is usually called the **induction step** of the proof. It is a good idea to point out when this occurs.

Doing proofs by induction can be a little unsettling. In using $P(k)$ as a *hypothesis*, it might look like we're assuming what we're trying to prove, an activity we've been warned about at some length. Fortunately, this is not the case. The thing we're trying to prove is a list of statements: $P(1), P(2), \ldots$, while the thing we're assuming is just one entry in the list. In any event, we are *not* assuming that $P(k)$ is true, we are only showing that *if* $P(k)$ is true, then so is $P(k+1)$.

4.4 ORGANIZING PROOFS BY INDUCTION

Some of the mystery can be taken out of induction by organizing the proofs carefully. It is useful to begin by writing down and labeling the open statement, and showing the variable clearly. For instance, we might wish to prove that[3]

$$P(n) : 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Now write down and check $P(1)$:

$$P(1) : 1 = \frac{(1)(2)(3)}{6}.$$

Write the induction hypothesis, labeling it "Assume" (or something else to indicate its role in the problem):

$$\text{Assume } P(k) : 1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

Write the *conclusion* of the inductive step, labeling it something like "Want":

$$\text{Want } P(k+1) : 1^2 + 2^2 + \cdots + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}.$$

Now the work begins. Very often (especially with arithmetic formulas), we can see something resembling the induction hypothesis within the

---

[3] The colon in this line only serves to separate the *name* of $P(n)$ from the *statement* of $P(n)$. One must be careful not to put an "=" here.

induction conclusion. In this problem it's not difficult to find. Notice that we haven't changed anything from the first line of this proof to the second, we have only highlighted a term that was already there.

$$\begin{aligned}
& 1^2 + 2^2 + \cdots + (k+1)^2 \\
= \; & 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 \\
= \; & \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{[induction hypothesis]} \\
= \; & (k+1)\frac{k(2k+1) + 6(k+1)}{6} \\
= \; & (k+1)\frac{2k^2 + 7k + 6}{6} \\
= \; & \frac{(k+1)(k+2)(2k+3)}{6} \quad \blacksquare
\end{aligned}$$

4.5 STRONG INDUCTION

Consider the collection of numbers $\{f_1, f_2, \ldots\}$ given by letting $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n = 3, 4, \ldots$. It seems clear enough that these are all natural numbers.[4] Each is found by adding two things that were in turn obtained by adding two things that (and so on and so on) were both natural numbers. Does this constitute a proof? We should be very suspicious of arguments that involve "and so on and so on" as a crucial step. We can recognize this as an induction problem:

$$\begin{aligned}
& P(n) : f_n \in \mathbf{N} \\
& P(1) : f_1 = 1 \in \mathbf{N} \qquad [P(2) \text{ is also true}] \\
& \text{Assume } P(k) : f_k \in \mathbf{N} \\
& \text{Want } P(k+1) : f_{k+1} \in \mathbf{N}
\end{aligned}$$

But now the going gets a little rocky. The value of f_{k+1} depends not just on f_k but on both f_k and f_{k-1}. If we can't find a formula for f_{k+1} depending *only* on f_k, we would seem to be stuck (you are welcome to look for such a formula). We can get around this by choosing P more carefully:

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[4] These are called the **Fibonacci numbers**. The first few are 1, 1, 2, 3, 5, 8, 13, ... These numbers are of great interest in both serious and recreational number theory and pop up with eerie regularity in descriptions of natural processes.

$$
\begin{aligned}
&P(n) : f_1, f_2, \ldots, f_n \in \mathbf{N} \\
&P(1) : f_1 = 1 \in \mathbf{N} \\
&\text{Assume } P(k) : f_1, f_2, \ldots, f_k \in \mathbf{N} \\
&\text{Want } P(k+1) : f_1, f_2, \ldots, f_k, f_{k+1} \in \mathbf{N}
\end{aligned}
$$

Now it's easy (we actually have more information than we need). Simply note that $f_{k+1} = f_k + f_{k-1}$, both of which are integers. ∎

This is an outline of a procedure called **strong induction** (though we will see that this is somewhat of a misnomer), which is described in the following theorem.

THEOREM 4.6: *Induction is equivalent to the following: Let $P(n)$ be an open statement, where n can be any natural number. If*

(i) *$P(1)$ is true*

and (ii) *$(P(1), \ldots,$ and $P(k)) \Rightarrow P(k+1)$ whenever $k \geq 1$,*

then $P(n)$ is true for all $n \in \mathbf{N}$.

We will show only that this statement implies induction. (This direction of the proof is of the form "weaker ⇒ stronger," and so it is the only one about which there is any real doubt.) While what is actually going on in this proof is not difficult, the structure of the proof is a little complicated. It is of the form $(A \Rightarrow B) \Rightarrow (C \Rightarrow B)$, where A consists of the hypotheses of Theorem 4.6, B is the statement "$P(n)$ is true for all n," and C consists of the hypotheses of induction (Theorem 4.5). The contrapositive of this is (not $(C \Rightarrow B) \Rightarrow$ not $(A \Rightarrow B)$), or ((C and not B) $\Rightarrow$ (A and not B)). This is equivalent to $C \Rightarrow A$, which is what we will prove.

PROOF: Suppose that $P(n)$ is an open statement and that the hypotheses of induction hold, that is, $P(1)$ is true and $P(k) \Rightarrow P(k+1)$ for $k \geq 1$. Then the first hypothesis of Theorem 4.6—$P(1)$ is true—holds. Suppose that $P(1), \ldots, P(k)$ are true. Then in particular, $P(k)$ is true. Since the hypotheses of induction hold, it follows that $P(k+1)$ is true, and we are done. ∎

Strong induction is important not so much as a separate technique of proof (by constructing our propositions carefully, we can avoid using it explicitly), but as a signpost to bigger and better things. If we rephrase strong induction in the language we first used to describe induction itself, it would look like this:

If $S \subseteq \mathbf{N}$ is such that $1 \in S$ and, for each $n > 1$,
$\{k : k < n\} \subseteq S \Rightarrow n \in S$, then $S = \mathbf{N}$.

Notice that this statement makes sense with **N** replaced by *any* well-ordered set and 1 replaced by the least element of the set (and there are well-ordered sets that are bigger and more complicated than we can possibly imagine just now). The resulting statement is a very deep and powerful tool called **transfinite induction**.

EXERCISES 4.5

1. Carefully state the definition of well-ordering in terms of the definition of an ordering based on sets of ordered pairs.
2. Show that the two parts of Definition 4.3 are equivalent (that is, if $m < n$ according to one definition, then $m < n$ according to the other).
3. Verify the claim that $\{p \in \mathbf{Q} : p > 0\}$ does not have a least element.
4. Show that the real numbers, rational numbers, and integers are not well-ordered in their usual orderings.
5. What is the smallest natural number that can be written as a sum of three primes but can't be written as a sum of two primes?
6. Construct a proof of Theorem 2.4 based on the fact that every finite set has a largest element.
7. Carefully state and prove that "*Induction implies well-ordering.*"
8. (a) Explain the bad example of induction.

   (b) Explain why the example just preceding it—in which much the same thing seems to be done—is valid.
9. (a) Explain why "strong induction" is a *weaker* theorem than induction.

   (b) Show that $((X \Rightarrow Z) \Rightarrow (Y \Rightarrow Z)) \Leftrightarrow (Y \Rightarrow X)$.
10. (a) Show that if S is a set with n elements, then the power set of S has 2^n elements (this is why it is called the power set).

    (b) Show that $n < 2^n$ for all $n \in \mathbf{N}$.
11. (a) Show that the following expression is an integer for $n = 0, 1, \ldots$

$$\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

    (It might help to do some experiments with a calculator first.)

(b) If x_n is the expression in (a), show that

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{1+\sqrt{5}}{2}.$$

This number is sometimes called the "golden ratio." It was considered by the ancient Greeks to represent a perfect proportion, and much of their art and architecture include shapes that contain it.

12. Let $P(n)$ be the statement "$n^2 + 9n + 5$ is even."

    (a) Show that $P(k) \Rightarrow P(k+1)$ for $k \geq 1$.

    (b) For which n is $P(n)$ true?

    (c) What went wrong?

13. (a) Show that any subset of an initial segment of **N** is finite. (Be very careful with the definitions here—this is not as simple as it looks. The word "finite" is a clue.)

    (b) Show that a subset of a finite set is finite.

14. (a) Prove this useful variation on induction: *If $P(n)$ is an open statement whose domain is* **Z** *and if*

    (i) $P(n^\star)$ is true for some $n^\star$
    and (ii) $P(k) \Rightarrow P(k+1)$ whenever $k \geq n^\star$,

    then $P(n)$ is true for all $n \geq n^\star$.

    (b) Find all values of n for which $n^3 < 2^n$. Prove your result.

15. (a) Suppose that S is a subset of **N** with the properties:

    (i) $2^n \in S$ for $n = 1, 2, \ldots$.
    and (ii) If $k \in S$ and $k > 1$, then $k - 1 \in S$

    Show that $S = \mathbf{N}$.

    (b) Condition (i) may be phrased "$P = \{2^n : n = 1, 2, \ldots\} \subseteq S$." State a *condition on* (as opposed to a *description of*) the set P that would yield the same result.

    (c) Could the condition "$k - 1 \in S$" be replaced by "$k - 2 \in S$" and keep the result?

16. (a) If $x_1, x_2, \ldots, x_n$ are all nonnegative, their **geometric mean** is defined to be $G(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}$ (note that, strictly speaking, G is not a single function but a collection of functions). Show that G is similar to the ordinary (algebraic) mean in the sense that

    (i) $\min\{x_1, x_2, \ldots, x_n\} \leq G(x_1, x_2, \ldots, x_n) \leq \max\{x_1, x_2, \ldots, x_n\}$.

    (ii) There is equality on either side above if and only if $x_1 = \cdots = x_n$.

(iii) $G(x_1, x_2, \ldots, x_n, G(x_1, x_2, \ldots, x_n)) = G(x_1, x_2, \ldots, x_n)$.

(First show that statements like these hold for the algebraic mean, then show them for the geometric mean.)

(b) Show that, if $x_1, x_2, \ldots, x_n$ are all nonnegative, then

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \leq \sqrt[n]{x_1 x_2 \ldots x_n}.$$

(This is called the **algebraic-geometric mean inequality**.)

17. Recall that the binomial coefficients are given by: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

(a) Show that binomial coefficients satisfy the equation

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

(b) This equation is closely related to Pascal's triangle. Explain.

(c) Verify the claim made in Exercise 2.2.7: The coefficient of x^k in $(1+x)^n$ is $\binom{n}{k}$.

18. (a) Show that $1 + 3 + \cdots + (2n-1) = n^2$ for $n = 1, 2, \ldots$.

(b) Show that $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ for $n = 1, 2, \ldots$.

19. If $a \geq -1$, show that $(1+a)^n \geq 1 + na$. This is called **Bernoulli's inequality**.

20. Show that $7^n - 6n - 1$ is divisible by 36 for $n = 1, 2, \ldots$.

21. (a) Show that, for any natural number n,

$$(1 \times 2) + (2 \times 3) + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

(b) Show that, for any natural number n,

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

(c) Do some experiments and make a guess of a formula for the sum of the first n fourth powers. Prove your result. (Is there a pattern in the formulas for the sums of first, second, and third powers? It is clearly a polynomial in the variable n. What is its degree? Such a formula is called a "closed form" for the sum.)

22. Bees have an unusual biology. A male bee has only one parent (the queen), while female bees have two parents. Starting with a male bee, count the number of its parents, grandparents, great-grandparents, and so on.

23. (a) Suppose that for each $m, n \in \mathbf{N}$, $P(m,n)$ is an open statement with variables m and n. Establish "double induction": *If*

$$\begin{aligned} &\text{(i)}\ \ P(1,1) \textit{ is true,} \\ &\text{(ii)}\ \ P(m,k) \Rightarrow P(m,k+1) \textit{ for any } m \textit{ and } k = 1,2,\ldots, \\ \textit{and}\ \ &\text{(iii)}\ \ P(k,n) \Rightarrow P(k+1,n) \textit{ for any } n \textit{ and } k = 1,2,\ldots, \end{aligned}$$

then $P(m,n)$ *is true for all* m *and* n.

(b) Show that $2^m 2^n = 2^{m+n}$ for all natural numbers m and n.

24. (a) Suppose $A = \{a_1, a_2, \ldots, a_n\}$ is a finite set of real numbers. Show that it need not be the case that $a_1 < a_2 < \cdots < a_n$ (a simple example will do).

(b) Show that such a set *can* be renumbered in such a way that $a_1 < a_2 < \cdots < a_n$ (that is, show that the function $f : \{1, \ldots, n\} \to A$ in the definition of finite can be taken to be increasing).

(c) Suppose S is a denumerable subset of the real numbers. Can the function $f : \mathbf{N} \to S$ that establishes that S is denumerable always be taken to be increasing? (That is, can a denumerable set always be numbered in increasing order?)

25. Show that $|\sin nx| \le n|\sin x|$ for all x and $n = 1, 2, \ldots$.

26. If $x_1, x_2, \ldots, x_n$ are all positive, their **harmonic mean** is:

$$H(x_1, x_2, \ldots, x_n) = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}.$$

(H is the reciprocal of the average of the reciprocals.)

(a) Show that the harmonic mean satisfies conditions (i), (ii), and (iii) of Exercise 4.5.16.a.

(b) By doing some experiments, guess an arithmetic-harmonic mean inequality. Prove your result.

(c) How about a harmonic-geometric mean inequality?

4.6 ORDERED FIELDS

We had to know a good bit about the natural numbers to define the ordering on them. It would be an enormous undertaking to specify the truth value of $a < b$ for all pairs of real numbers. Furthermore, we have a secondary goal to consider. Whatever ordering we devise for the real numbers should be closely related to their field structure. Though this consideration may seem to make the problem more difficult, it actually allows us to narrow our focus. Since we can *subtract* in a field, it isn't

necessary to assign a truth value to $a < b$ for every a and b, but only to the expression $x > 0$ for each x (we would certainly want $a < b$ to have the same truth value as $b - a > 0$).

DEFINITION 4.7: (a) Let $\mathbf{P}$ be a nonempty subset of a field $\mathbf{F}$. Suppose

(i) If $a \in \mathbf{P}$ and $b \in \mathbf{P}$, then $ab \in \mathbf{P}$ and $a + b \in \mathbf{P}$

and (ii) For each $x \in \mathbf{F}$, exactly one of $x \in \mathbf{P}, x = 0$, or $-x \in \mathbf{P}$ holds.

Then $\mathbf{P}$ is called a **positive set**.

(b) A pair $(\mathbf{F}, \mathbf{P})$, where $\mathbf{F}$ is a field and $\mathbf{P}$ is a positive set, is called an **ordered field**.

(c) In an ordered field, we say $\boldsymbol{a<b}$ if $b + (-a) \in \mathbf{P}$.

If $x \in \mathbf{P}$, we say x is **positive**, and if $-x \in \mathbf{P}$, we say x is **negative**. Note that the ordering on an ordered field is linear. This definition is related to the field structure in a big way. The only substantial part of the definition of a field we don't see is the multiplicative inverse.

A precise definition of a positive set in the real numbers is given in Chapter 22. For now, we will have to view statements about the positive set in the real numbers as we have statements about the field structure of the real numbers: We know that they will be proved later. Suffice it to say that the positive set of the real numbers is pretty much what we think it is. We spent years in elementary school looking at theorems about the ordering of the real line. Here we will select a few that are of particular importance and that illustrate the interplay of algebra and order.

THEOREM 4.8: *If $(\mathbf{F}, \mathbf{P})$ is an ordered field, $a \in \mathbf{F}$, and $a \neq 0$, then $a^2 \in \mathbf{P}$.*

PROOF: Since $a \neq 0$, either $a \in \mathbf{P}$ or $-a \in \mathbf{P}$. In the first case, the result follows from the definition of $\mathbf{P}$. In the second case, note that $a^2 = (-a)^2 \in \mathbf{P}$ (by Exercise 3.2.11). ∎

COROLLARY 4.9: *In an ordered field, $1 \in \mathbf{P}$.*

PROOF: Since $P \neq \emptyset$ and $0 \notin \mathbf{P}$, an ordered field has more than one element. By Exercise 3.2.9, we have $1 \neq 0$, and so $1 = 1^2 \in \mathbf{P}$. ∎

THEOREM 4.10: *The product of a positive element of an ordered field and a negative element is negative.*

PROOF: If $a \in \mathbf{P}$ and $-b \in \mathbf{P}$, then $-(ab) = a(-b) \in \mathbf{P}$. ∎

COROLLARY 4.11: *In an ordered field, $x \in \mathbf{P}$ if and only if $x^{-1} \in \mathbf{P}$.*

PROOF: Left as Exercise 4.6.5. ∎

We can use these results to show that the real numbers and the complex numbers are different: By Corollary 4.9, we have $-1 \notin \mathbf{P}$ in any ordered field. In $\mathbf{C}$, though, -1 is a square ($-1 = i^2$), and so by Theorem 4.8, we should have $-1 \in \mathbf{P}$, a contradiction. We have not only shown that our usual idea of ordering doesn't work in $\mathbf{C}$. We've shown that it is *impossible* to define an ordering that makes $\mathbf{C}$ into an ordered field.

Most of what we learned in our youth about the ordering of the real numbers is contained in the next theorem.

THEOREM 4.12: *Let a, b and c be elements of an ordered field.*
(a) *If $a < b$, then $a + c < b + c$.*
(b) *If $a < b$ and $c \in \mathbf{P}$, then $ac < bc$.*
(c) *If $a < b$ and $-c \in \mathbf{P}$, then $bc < ac$.*
(d) *If $a < b$, then $a < (a + b)/2 < b$.*
(e) *If $a < b$ and $b < c$, then $a < c$.*

PROOF: We will prove parts (a) and (d). (a) We want $(b+c)-(a+c) \in \mathbf{P}$. But $(b + c) - (a + c) = b - a \in \mathbf{P}$ by hypothesis.
(d) Using part (a) twice, we have: $2a = a + a < a + b < b + b = 2b$. Observe that $2 = 1 + 1 \in \mathbf{P}$. By Corollary 4.11, we may divide by 2 to obtain the result. ∎

So far, we have tended to consider the natural numbers as something apart from the other structures we've discussed. We can bring the discussions of the natural numbers and ordered fields together by observing that every field contains the elements: $1, 1 + 1, 1 + 1 + 1, \ldots$. In some fields, these elements aren't all different (in $\mathbf{Z}_2$, for instance, we have $1 + 1 + 1 = 1$). The next theorem tells us that this can't happen in an ordered field, and consequently that every ordered field contains a "copy" of $\mathbf{N}$.

THEOREM 4.13: *In an ordered field, the elements $1, 1+1, 1+1+1, \ldots$ are all positive and all different.*

PROOF: Consider the open statement $P(n) : \left(\sum_{j=1}^{n} 1\right) \in \mathbf{P}$. Notice that the 1 that is the lower limit of the summation is a natural number, while the 1 that is inside the summation is the multiplicative identity of the field.

$$P(1) : 1 \in \mathbf{P} \text{ is true} \quad \text{[Corollary 4.9]}$$

$$\text{Assume } P(k) : \left(\sum_{j=1}^{k} 1\right) \in \mathbf{P}.$$

$$\text{Want } P(k+1) : \left(\sum_{j=1}^{k+1} 1\right) \in \mathbf{P}.$$

Now $\sum_{j=1}^{k+1} 1 = \left(\sum_{j=1}^{k} 1\right) + 1$, and so $\sum_{j=1}^{k+1} 1 \in \mathbf{P}$ since it is the sum of two elements of $\mathbf{P}$. The result follows by induction. To see that these elements are all different, observe that the difference of any two of them is either another one of them or the additive inverse of one of them. Any such difference is either positive or negative (that is, *not* 0). ∎

COROLLARY 4.14: *An ordered field is infinite.* ∎

We may refer to the elements $1+1, 1+1+1, \ldots,$ of an ordered field as $2, 3, \ldots.$ Note that a field that contains such a copy of $\mathbf{N}$ also contains copies of $\mathbf{Z}$ and $\mathbf{Q}$, and so it makes sense to refer to "integers" and "rational elements" in any ordered field.

EXERCISES 4.6

1. Show that the ordering on an ordered field is linear.

2. (a) Suppose a and b are elements of an ordered field. Show that $a^2 + b^2 = 0$ if and only if $a = b = 0$.

   (b) Does this remain true if the field is not assumed to be ordered?

3. (a) None of the fields $\mathbf{Z}_p$ can be ordered (why?). Give an example of a value of p and elements a and b of $\mathbf{Z}_p$ such that $a \neq 0$ and $b \neq 0$ but $a^2 + b^2 = 0$.

   (b) Are there values of p for which the situation in (a) *can't* happen? Consider a characterization of all values of p for which this is possible (or impossible).

4. In the complex numbers, we say the distance from $a + bi$ to 0 is $\sqrt{a^2 + b^2}$. We might define an ordering on $\mathbf{C}$ by saying $z < w$ if it is closer to 0. Show that this *does not* make $\mathbf{C}$ into an ordered field.

5. Prove Corollary 4.11.

6. Complete the proof of Theorem 4.12.

7. Complete the induction in the proof of Theorem 4.13.

8. Show that the field of rational numbers has the following property: Given any rational number $r = p/q$, there is a natural number n_r with $r < n_r$. (This is not difficult. Simply construct n_r from r.) An ordered field in which the natural numbers are distributed in this way is said to have the **Archimedean property**. This will be carefully defined and discussed in Chapter 6, where we will show that it also holds for the real numbers.

9. Show that $\{r : r = p/q,$ where $p, q \in \mathbf{Z}$ have the same sign$\}$ is a positive set on $\mathbf{Q}$.

10. (a) Show that $\{x \in \mathbf{R} : \exists p, q \in \mathbf{Q} \ni (p, q > 0 \text{ and } x \in [p, q])\}$ is a positive set on $\mathbf{R}$.

    (b) Explain why this couldn't be used to *define* a positive set on $\mathbf{R}$.

11. Consider the field $\mathbf{M}_2$ defined in Exercise 3.2.14.

    (a) We might try to define a positive set on $\mathbf{M}_2$ by saying one of these matrices is to be "positive" if both a and b are positive. Show that this *doesn't* make $\mathbf{M}_2$ into an ordered field.

    (b) What, if anything, does Exercise 3.2.14.c say about the possibility of defining a positive set on $\mathbf{M}_2$?

12. (a) Show that in $\mathbf{Z}_5$ we have $4 = -1$ (remember what -1 means!).

    (b) Find an element of $\mathbf{Z}_5$ that satisfies the equation $x^2 = -1$.

    (c) What does part (b) say about the possibility of making $\mathbf{Z}_5$ into an ordered field?

    (d) But isn't $0 < 1 < 2 < 3 < 4$ a perfectly good ordering on $\mathbf{Z}_5$? Explain.

13. Suppose a and b are real numbers such that for every $\varepsilon > 0, a + \varepsilon > b$. Show that $a \geq b$. Show that it is not necessarily the case that $a > b$.

14. (a) Explain why a field that contains a copy of $\mathbf{N}$ also contains copies of $\mathbf{Z}$ and of $\mathbf{Q}$.

    (b) Show that $\mathbf{C}$ contains copies of $\mathbf{N}$, $\mathbf{Z}$, and $\mathbf{Q}$ (identify them exactly). Nevertheless, $\mathbf{C}$ cannot be ordered. Is there a contradiction here?

    (c) The set of integers contains a perfectly good copy of $\mathbf{N}$, but does not contain a copy of $\mathbf{Q}$. How can this be?

15. Can a field be ordered in more than one way? That is, can there be two sets $P_1 \neq P_2$ that both satisfy the definition of a positive set?

4.7 ABSOLUTE VALUE AND DISTANCE

Since each nonzero element of an ordered field is either positive or negative, we may make the following definition.

DEFINITION 4.15: The **absolute value** of an element x of an ordered field $(\mathbf{F}, \mathbf{P})$ is given by:

$$|x| = \begin{cases} x & \text{if } x \in \mathbf{P} \text{ or } x = 0 \\ -x & \text{if } -x \in \mathbf{P} \end{cases}$$

It doesn't matter whether we define $|0|$ to be 0 or -0. The absolute value function is familiar from algebra and calculus. We will state only one theorem, leaving its proof as Exercise 4.7.1 (good practice in proof by cases):

THEOREM 4.16: (The Triangle Inequality) *If x and y are elements of an ordered field then*

$$\big||x| - |y|\big| \leq |x + y| \leq |x| + |y|. \blacksquare$$

DEFINITION 4.17: If x and y are elements of an ordered field, the **distance between x and y** is given by $|x - y|$.

The measurement of distances is crucial to our study, since we are often concerned whether points are close together. We may restate the right half of the Triangle inequality in this way: *If x, y, and z are elements of an ordered field, then* $|x - y| \leq |x - z| + |z - y|$ (this is the symbolic formulation of the phrase "The shortest path between two points is a straight line").

We can begin to see the difference between two different approaches to analysis. "Hard" analysis is greatly concerned with the proof and application of inequalities, while "soft" analysis (the kind we will be doing) is not. The Triangle inequality is about the only inequality we will need.

EXERCISES 4.7

1. (a) Prove the Triangle inequality.

   (b) Under what circumstances can the $\leq$ signs in the Triangle inequality be replaced with $=$ signs?

   (c) Show that $|-a| = |a|$.

   (d) Show that $|ab| = |a||b|$ and that $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ as long as $b \neq 0$.

   (e) Show that $|a| = \sqrt{a^2}$ for all a.

2. Use the left side of the triangle inequality to show that addition is continuous in the following sense. For any $a \in \mathbf{R}$, Define the function f_a by: $f_a(x) = a + x$. Then f_a is continuous for any a.

3. If we let $d(x, y) = |x - y|$, show that d is a **metric** in the sense of linear algebra, that is:
   (i) $d(x, y) \geq 0$ for all x, y, and $d(x, y) = 0$ if and only if $x = y$.
   (ii) $d(x, y) = d(y, x)$
   and (iii) $d(x, z) \leq d(x, y) + d(y, z)$

4. If $\vec{x} = (x_1, x_2) \in \mathbf{R}^2$, let $D_1 : \mathbf{R}^2 \to \mathbf{R}$ and $D_2 : \mathbf{R}^2 \to \mathbf{R}$ be given by
$$D_1(\vec{x}, \vec{y}) = |y_1 - x_1| + |y_2 - x_2|$$
and
$$D_2(\vec{x}, \vec{y}) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}.$$
Show that D_1 and D_2 are both metrics on $\mathbf{R}^2$.

4.8 INTERVALS

Linearly ordered sets have a very useful type of subset called *intervals*. We are, of course, only interested in intervals of real numbers and (on rare occasions) of rational numbers.

DEFINITION 4.18: Let S be a linearly ordered set. The set $I \subseteq S$ is an **interval** if I has one of the following forms:[5]

(1)	$\{x \in S : a < x < b\}$	denoted (a, b)
(2)	$\{x \in S : a \leq x \leq b\}$	$[a, b]$
(3)	$\{x \in S : a \leq x < b\}$	$[a, b)$
(4)	$\{x \in S : a < x \leq b\}$	$(a, b]$
(5)	$\{x \in S : a < x\}$	(a, ∞)
(6)	$\{x \in S : x < b\}$	$(-\infty, b)$
(7)	$\{x \in S : a \leq x\}$	$[a, \infty)$
(8)	$\{x \in S : x \leq b\}$	$(-\infty, b]$
(9)	all of S	(sometimes) $(-\infty, \infty)$

[5] We usually assume that $a < b$ unless we specifically state otherwise. However, if $a = b$ then $[a, b] = \{a\}$, and if $b \leq a$ we have $(a, b) = \emptyset$, both of which make sense. This means that $\emptyset$ and a set consisting of a single point *are* intervals.

The symbol ∞ should not be endowed with any meaning except that given it here. It is particularly important to remember that this symbol does *not* represent an element of S. We insist that S be linearly ordered mainly because the sets that are of interest to us are, though the linear ordering does allow us to write $S = (-\infty, \infty)$. We are concerned primarily with intervals of the first and second types, called **open** and **closed**, respectively. Sets of the forms (5), (6), (7), and (8) are sometimes called **rays**.

Open intervals play a crucial role in defining an idea of "nearness" on the real line. Closed intervals will attract our attention less often, but when they do, it will be in a pivotal way. Here we will establish some technical results. These are chosen not so much for their depth but because they will be useful later. They are stated for intervals of the form (a, b) but can be modified to apply to intervals of other types. The first of these concerns the relationship between intervals and the measurement of distance in the real numbers and gives us a hint how we might use intervals to define nearness.

THEOREM 4.19: *If x and y are elements of (a, b), then $|x - y| < b - a$.*

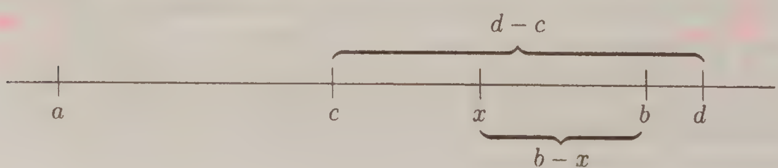
PROOF: We may suppose that $a < y \leq x < b$, so that $|x - y| = x - y$. Then $x - y < b - y < b - a$, by Theorem 4.12. ∎

THEOREM 4.20: *If $x \in (a, b) \cap (c, d)$, and $d - c < \min\{b - x, x - a\}$, then $(c, d) \subseteq (a, b)$.*

PROOF: Let $y \in (c, d)$ and suppose that $x < y$. By Theorem 4.19, $y - x = |x - y| < d - c < \min\{b - x, x - a\} \leq b - x$. Thus $a < x < y < b$, and so $y \in (a, b)$, as desired. A similar argument holds if $y < x$. ∎

4.9 WHEN SHOULD WE DRAW PICTURES?

Though the proof of Theorem 4.20 is brief, it may not be entirely clear what is going on in it, or what it is really about. This can be cleared up with a few pictures. First, we look at a situation where the result does not hold:



and one where it does:

Note that $\min\{b-x, x-a\}$ is the distance from x to the nearer endpoint of (a, b). In the second picture, the inner interval isn't long enough to reach either end of the outer interval. This is what Theorem 4.20 is about.

How useful are pictures? We know that we generally can't use them to prove things, but they can certainly help us get ideas. We could have introduced Theorem 4.20 with a question: "Give a condition that will guarantee that $(c, d) \subseteq (a, b)$." After drawing a few sketches like these, we would arrive at something like Theorem 4.20. We should never hesitate to draw sketches but must keep in mind their limitations. The main object of our study, the real number line, is not exactly the stuff of striking artwork, and so all our pictures will look very much like these two.

More importantly, *when* should we draw a picture? You will notice that proofs almost never *start* with a picture. How do we decide when it is time to draw one? There are, as you might guess, no rules to help us with this, but there is one very useful guideline: Draw a picture when you have something to draw! When you reach an *existentially* quantified step in a proof (that is, when you are sure that something with specific properties exists), draw it, but be careful to include in your picture only those features that your current knowledge allows.

EXERCISES 4.9

1. (a) Show that, if a and b are elements of a linearly ordered set S and $b > a$, then $(a, \infty) \cup (-\infty, b) = S$.

   (b) Show that this does not necessarily hold if the ordering of the set is not linear. This is why we need S to be linearly ordered to say that $S = (-\infty, \infty)$.

2. (a) Show that if $c < d$ and (a, b) contains neither c nor d, then either $(a, b) \cap (c, d) = \emptyset$ or $(a, b) \subseteq (c, d)$.

   (b) Suppose that A and B are open intervals such that neither contains an endpoint of the other. Show that A and B are either disjoint or identical. (This result will play an important role in the proof of Theorem 8.11.)

3. Suppose S is a set with the property that $|x - y| \geq 1$ for any two different elements x and y of S. Show that an interval $(a, a+1)$ can contain at most one element of S. Does this result change if a half-open or closed interval is used instead of an open interval?

4. (a) If $I = (a, b)$ and $J = (c, d)$ are open intervals, show that $I \subseteq J$ if and only if $a \geq c$ and $b \leq d$.

   (b) Does this result change if the intervals are not open?

5. (a) Show that the intersection of two intervals is an interval.

   (b) If I and J are intervals with $I \cap J \neq \emptyset$, show that $I \cup J$ is an interval.

   (c) Is the condition "$I \cap J \neq \emptyset$" necessary in (b)?

   (d) Fill in the blank: If I and J are intervals, then $I \cup J$ is an interval is and only if ____________________.

   (e) Reinterpret (a) through (d) with "interval" replaced by "ray."

6. Modify Theorems 4.19 and 4.20 to allow for intervals that are not open.

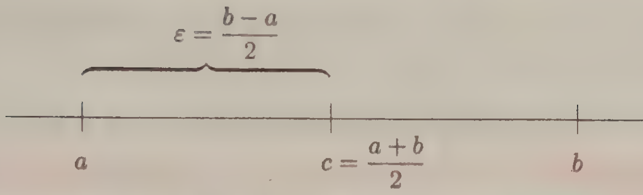
7. In the proof of Theorem 4.19, why can we suppose that $y \leq x$?

4.10 NEIGHBORHOODS

The next theorem gives us a useful way of describing intervals. Instead of giving the endpoints of an interval, we may specify a "center" and a "radius." We will often be concerned with intervals that are, so to speak, "short," and the ε in the following theorem is an easy way of measuring this. We will consider two points to be close together if they are in many of the same short intervals. This theorem also shows one way in which the order and distance-measuring (or "metric") structures of the real line are related.

THEOREM 4.21: *If $a < b$, let $c = (a+b)/2$ and $\varepsilon = (b-a)/2$. Then*
$$(a, b) = \{x : |x - c| < \varepsilon\}.$$

PROOF: It seems that c is exactly in the middle of a and b, and that ε is half the distance from a to b (these guesses are confirmed in the proof):



Observe first that this is a set-equality problem, and so the structure of the proof is determined for us. Let $x \in (a,b)$ and suppose $x \geq c$. Then[6] $x \in [c,b)$, and so $|x-c| < b-c = (b-a)/2 = \varepsilon$. Thus $x \in \{t : |t-c| < \varepsilon\}$ (and similarly if $x < c$). Now suppose $x \in \{t : |t-c| < \varepsilon\}$ and $x \geq c$. Then $|x-c| = x-c < \varepsilon$, and so $a < c \leq x < c+\varepsilon = b$, and so $x \in (a,b)$ (and similarly if $x < c$). ∎

DEFINITION 4.22: An interval of the form $\{x : |x-c| < \varepsilon\}$ for some real number c and some positive real number ε is called an **ε-neighborhood of c** (or an **ε-interval around c**).

An ε-neighborhood consists, roughly, of all points near its center. This is such a useful property that we will often forgive a set for containing other points. This spirit of forgiveness leads us to the following definition.

DEFINITION 4.23: The set U is a **neighborhood** of c if there exists $\varepsilon > 0$ so that U contains the ε-neighborhood of c.

Note that every ε-neighborhood of a point is a neighborhood of it, but *not every neighborhood is an ε-neighborhood.* We can use an ε-neighborhood as an example in an *existential* proof, but in a *universal* proof we may *not* assume that a set identified only as a neighborhood is necessarily an ε-neighborhood.

EXAMPLES 4.10: 1. $(0,1)$ is a neighborhood of $1/4$, but is not an ε-neighborhood of $1/4$. The $1/8$-interval around $1/4$ is contained in $(0,1)$. Note that any $\varepsilon \leq 1/4$ will work in this argument, but since the definition of neighborhood is existential, we only need to find one such value. The interval $(0,1)$ *is* an ε-neighborhood of $1/2$, with $\varepsilon = 1/2$.

2. The closed interval $[0,1]$ is also a neighborhood of $1/4$ (by the same argument as above) but it is not a neighborhood of 0. Any ε-neighborhood of 0 contains negative numbers ($-\varepsilon/2$, for one) and so is not contained in $[0,1]$. If a set is a neighborhood of a point, the point must be an element of the set, but *a set can contain points of which it is not a neighborhood.*

[6] We use a modification of Theorem 4.19 here.

3. The set $(0,1) \cup \{2,3,4,\dots\}$ is also a neighborhood of $1/4$, but is not a neighborhood of any of 2, 3, 4, ...

Example 1 is a special case of a very general result:

THEOREM 4.24: *An open interval is a neighborhood of each of its points.*

PROOF: Let $x \in (a,b)$ and $\varepsilon = \min\{x-a, b-y\}$. Then $(x-\varepsilon, x+\varepsilon) \subseteq (a,b)$ by Theorem 4.20. Since such an ε exists, (a,b) is a neighborhood of x. ∎

EXERCISES 4.10

1. Suppose U is a neighborhood of a point x and that $U \subseteq V$. Show that V is a neighborhood of x.
2. (a) Suppose U and V are neighborhoods of a point x. Show that $U \cap V$ and $U \cup V$ are neighborhoods of x.

   (b) Show that (a) remains true for a finite collection of neighborhoods.

   (c) Does (a) remain true for an infinite collection of neighborhoods?
3. Show that any set that is a neighborhood of some point is uncountable.
4. If $x \neq y$, show that there are neighborhoods U of x and V of y such that $U \cap V = \emptyset$.
5. Draw a sketch to illustrate the proof of Theorem 4.24.

Part Two

The Structure of the Real Number System

We have found that the real numbers and the rational numbers have much in common in that they are both ordered fields. We turn now to the question that is the central theme of the book: How are these two fields different? We saw in Chapter 2 that a certain set of decimals has greater cardinality than the set of rational numbers. Perhaps this settles the issue. There would seem to be *more* real numbers than there are rational numbers. But how are those decimals related to the real numbers? This is a tough question (we will answer it in Chapter 6).

The difference between the real and rational numbers is more than a mere counting problem. In this part of the book, we will be primarily concerned with six properties that the real numbers have but the rational numbers do not. This puts us in a peculiar position, which, though we should be aware of it, does not affect what we do. We will distinguish an ordered field having these properties from one not having them, and we will see that the rational numbers do not have these properties. But we will only *postulate* that there is an ordered field that does have them. Think of six descriptions of an imaginary beast. Any animal fitting one of your descriptions fits them all. A house cat, for instance, fits none of them. Even though you have clearly identified your animal as *not a cat*, it still does not exist! In the end, we can convince doubters by actually capturing one of our beasts. Western explorers of Australia vindicated themselves in just this way after years of having their tales of the platypus laughed at. We will find, though, that our study of the properties we will describe can proceed just as well whether or not we have a sample on hand (in this way, mathematics is not like biology). We will finally capture the real numbers in the last chapter of the book.

The properties of the real numbers we will discuss are found in the following theorem. **PLEASE REMAIN CALM!** Theorem **R**, as we will call it, is the concern of all of Part 2 of the book. We *shouldn't* know what these statements mean yet! (Notice that Theorem **R** includes

three "properties," two "theorems," one "criterion," and one statement without a fancy title. The differences among these names are historical accidents with no intellectual significance.)

THEOREM R: *If* **F** *is an ordered field having the Least Upper Bound property, then* **F** *has the Archimedean property and the following results also hold in* **F**. (The Least Upper Bound property is discussed in Chapter 5; the Archimedean property is discussed in Chapter 6.)

(a) *Every nest of closed, bounded intervals in* **F** *has a nonempty intersection.* (This is called the Nested Intervals property—Chapter 6.)

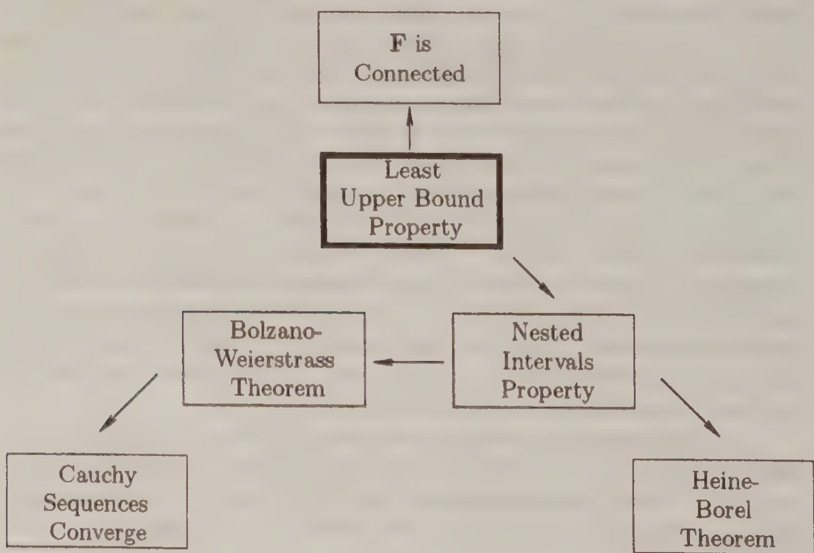
(b) *Every bounded, infinite subset of* **F** *has a cluster point.* (This is called the Bolzano-Weierstrass theorem—Chapter 7.)

(c) *A sequence in* **F** *converges to an element of* **F** *if and only if it is a Cauchy sequence.* (This is called the Cauchy criterion—Chapter 10.)

(d) *A subset of* **F** *is compact if and only if it is closed and bounded.* (This is called the Heine-Borel theorem—Chapter 11.)

(e) **F** *is connected.* (Chapter 12)

The relationships among the main parts of Theorem **R** are indicated in the following diagram. Each arrow represents one of our proofs (the Archimedean property is off to the side of this picture):



We denote the field in Theorem **R** by **F** because we are not certain that it is the real numbers as we usually think of them. The Nested Intervals property [part (a) of Theorem **R**] will tell us, among other things, that elements of such an ordered field have decimal expansions and that each decimal expansion corresponds to an element of the field. After proving this, we may confidently refer to the field as the real numbers.

COMPLETENESS AND THE BIG THEOREM

Theorem **R** does not tell the whole story of the real numbers. In the diagram above, it appears that everything grows from the Least Upper Bound property and that the Heine-Borel theorem, for instance, is not directly related to the Bolzano-Weierstrass theorem. But there is much more to the structure of the real numbers. The Least Upper Bound property and parts (a) through (e) of Theorem **R** are not just loosely related statements about the real numbers; they are *equivalent*,[2] that is, they describe the same property of the real numbers. This property is called **completeness**, which is, in a word, the answer to the Big Question. We may define the **real numbers** to be a *complete, Archimedean ordered field.* (We saw in Exercise 4.6.8 that the rational numbers also have the Archimedean property; it is completeness that makes the real numbers special.)

We refer to the following as the "Big Theorem" because it is the answer to the Big Question. The proof of the Big Theorem is the subject of most of Part 2 of the book.

THE BIG THEOREM: *If* **F** *is an ordered field, the following are equivalent:*

(a) **F** *has the Least Upper Bound property.*

(b) **F** *has the Archimedean property, and the Nested Intervals property.*

(c) **F** *has the Archimedean property, and the Bolzano-Weierstrass theorem holds in* **F**.

(d) *The Heine-Borel theorem holds in* **F**.

(e) **F** *has the Archimedean property, and the Cauchy criterion holds in* **F**.

(f) **F** *is connected.*

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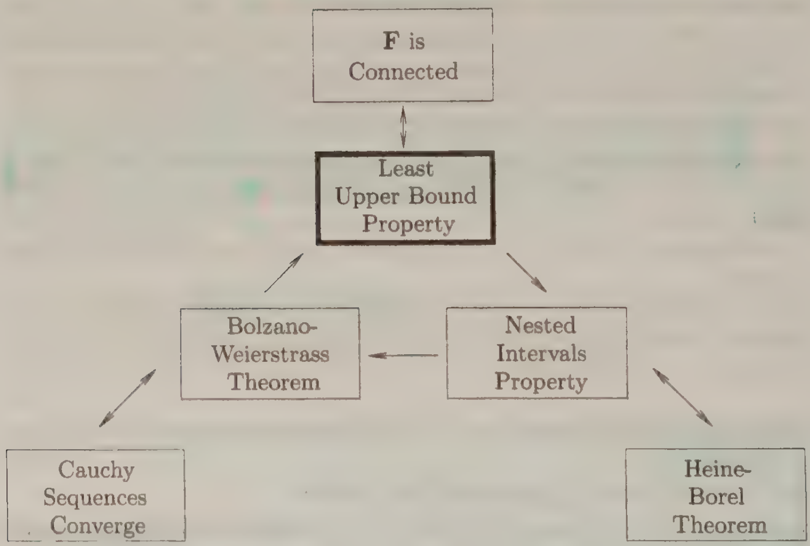
[2] Their equivalence is conditioned on the somewhat mysterious fluttering about of the Archimedean property. How and why the Archimedean property enters this story is a study in itself. We will confine our efforts to making sure that the Archimedean property is actually used when we claim it is used in our proofs, and not used when we claim it is not.

To show that these statements are equivalent, we must show that each of them implies each of the others. This would be thirty implications! Fortunately, we don't have to work quite that hard. A proof could be made with only six implications, like this: $a \Rightarrow b \Rightarrow c \Rightarrow d \Rightarrow e \Rightarrow f \Rightarrow a$. Our approach is not this economical, but if we can establish a chain of implications leading *from* each statement *to* each of the others, we will have shown that all are equivalent. Our proof of the Big Theorem is described in the following diagram, which we will call the Big Picture (as in "It is always important to see ..."):

THE BIG PICTURE



Since they are equivalent, we may take any of the six main statements of Theorem **R** to be the definition of completeness. Some authors refer to the Least Upper Bound property in this way (as will we, most of the time), while others use the word in specific reference to the Cauchy criterion. There are good reasons for both choices, and in fact we will find just once (in Chapter 15) that the latter usage serves us better.

The Big Theorem is important not so much because it asserts that each of the six characterizations of completeness is true (Theorem **R** tells us that), but because it asserts that they are equivalent. However, if one's goal is primarily to establish the *validity* of each of these statements, it is just as well to prove Theorem **R**, whose structure is not quite so elaborate. The proofs that transform Theorem **R** into the Big Theorem

are contained in sections entitled "Closing the Loop." These sections may be considered optional.

EXERCISES

1. Show that "If we can establish a chain of implications leading *from* each statement *to* each of the others, all of them are equivalent" in the following way:

   (a) Show that $((A \Rightarrow B)$ and $(B \Rightarrow C)) \Rightarrow (A \Rightarrow C)$.

   (b) Explain how this proves the statement.

Chapter 5

Upper Bounds and Suprema

5.1 UPPER AND LOWER BOUNDS

The ordering of our field allows us to define another important idea—that of an upper bound for a set. There are many possible relationships between a number and a set. The simplest of these is that a number might be an element of a set or it might not. The number 3 is not in the interval $[0, 1]$, but it is *larger* than everything in the set, and this tells us something, too. Many arguments rest on estimates of the size of the elements of a set.

DEFINITION 5.1: (a) The number u is an **upper bound** for the set S if $s \leq u$ for each $s \in S$. If S has an upper bound, we say that it is **bounded above**.

(b) The number w is a **lower bound** for the set S if $s \geq w$ for each $s \in S$. If S has a lower bound, we say that it is **bounded below**.

So 3 is an upper bound for $[0, 1]$, as are 113, 3/2, 11/10, 1, and infinitely many others. We see immediately that if a set has any upper bound, it has infinitely many of them. Not every set has an upper bound. Suppose I think the number u is an upper bound for the whole real line. You need only point out that $u + 1$ is a real number that is larger than u to show me that u is not larger than *every* real number and so u is not an upper bound for $\mathbf{R}$. The set of natural numbers is bounded below, since $n > 0$ for all $n \in \mathbf{N}$. The previous argument can be used to show that no *natural number* is an upper bound for $\mathbf{N}$. This does not exclude the possibility that there is some real number that is not a natural number but is an upper bound for $\mathbf{N}$ (be sure you see why this is so). We will show that $\mathbf{N}$ is not bounded above in the next chapter, but note how much machinery is necessary to accomplish this.

Among the upper bounds of the set $[0, 1]$, the number 1 is special. It is the only one for which there is no *smaller* upper bound. This seems obvious, but let's check it carefully. Suppose $u > 1$. By Theorem 4.12,

$1 < (1+u)/2 < u$, and so $(1+u)/2$ is an upper bound for $[0,1]$. But $(1+u)/2 < u$, and so there *is* an upper bound for $[0,1]$ that is less than u. Such a u does not have the special property we're looking for. On the other hand, if $u < 1$, then u can't be an upper bound for $[0,1]$ because $1 \in [0,1]$ and so u is less than an element of $[0,1]$. This special upper bound, and the analogous special lower bound, are given names in the following definition.

DEFINITION 5.2: (a) The number u is the **supremum** (or **least upper bound**) of the set S if

(i) u is an upper bound for S

and (ii) there is no upper bound for S less than u.

(b) The number w is the **infimum** (or **greatest lower bound**) of S if

(i) w is a lower bound for S

and (ii) there is no lower bound for S greater than w.

We will denote the supremum of a set S by $\sup S$ (if we say "least upper bound" we write lub S), and the infimum by $\inf S$ (or glb S).

EXAMPLES 5.1: 1. We have seen that $1 = \sup[0,1]$.

2. It is also true that $1 = \sup(0,1)$. Clearly 1 is an upper bound for this set. We must show that no smaller number is an upper bound, that is, for a given $v < 1$ we must find $t \in (0,1)$ with $t > v$. Since $1 \notin (0,1)$, the previous argument doesn't work. Note that $v < (v+1)/2 < 1$. Is $(v+1)/2$ the number we're looking for? It is true that $(v+1)/2$ is greater than v, but it is not necessarily in $(0,1)$. If $v = -11$, then $(v+1)/2 = -5$, which is not in $(0,1)$. On the other hand, if $v = -11$, the number $1/2$ will serve our purpose. Let

$$t = \begin{cases} \dfrac{v+1}{2} & \text{if } v \geq 0 \\ \dfrac{1}{2} & \text{if } v < 0. \end{cases}$$

Then $t > v$ and $t \in (0,1)$, and so v is not an upper bound for $(0,1)$. It follows that $1 = \sup(0,1)$.

This example also demonstrates the important fact that *the supremum of a set need not be in the set.* If we happen to know that the supremum of a set *is* an element of the set, we will call it the **maximum** and write $\max S$ instead of $\sup S$. When we refer to a number as the maximum of a set, we are making two claims that must *both* be verified: (1) That the number is the supremum of the set; and (2) that it is an element of the

set. If we are certain that the infimum of a set is an element of the set, we call it the **minimum** and write $\min S$ instead of $\inf S$. It is true that $1 = \max[0, 1]$, but *not* that $1 = \max(0, 1)$. "Maximum" and "minimum" are innocent-sounding words that must be used with caution.

3. Let $H = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. By Corollary 4.11 and Theorem 4.13, each element of H is positive, and so 0 is a lower bound for H. To show that 0 is the infimum of H, we must establish that there is no lower bound for H greater than 0. That is, we must show that $v > 0 \Rightarrow \exists n \in \mathbf{N} \ni (1/n < v)$. This appears to be so, but *we can't prove it yet*! We will prove it in Chapter 6.

4. The set $\{r \in \mathbf{Q} : r > 0 \text{ and } r^2 < 2\}$, thought of as a subset of $\mathbf{Q}$, has no rational supremum even though it is bounded above. This will be shown in detail in the proof of Theorem 5.4.

The phrase "u is an upper bound for S" may be written $\forall s \in S(u \geq s)$. Thus the statement "$v$ is not an upper bound for S" may be written: $\exists s \in S \ni (v < s)$. We may restate the definition of supremum:

$$u = \sup S \Leftrightarrow (\forall s \in S(s \leq u)) \text{ and } (v < u \Rightarrow \exists s \in S \ni (v < s))$$

(and a similar statement for the infimum). This makes the last part of the proof of the following theorem immediate. It will be helpful to have a variety of means by which to check that a number is the supremum of a set.

THEOREM 5.3: *u is the supremum of S if and only if, for any $\varepsilon > 0$, it is both the case that there is no element of S greater than $u + \varepsilon$ and that there is an element of S greater than $u - \varepsilon$.*

PROOF: That $u = \sup S$ implies the other conditions is clear. Suppose u satisfies the last two conditions of the theorem. We show first that such a u is an upper bound for S. Suppose $u < x$. Then $(x-u)/2 > 0$, and, by Theorem 4.12, $u < (u+x)/2 = u + (x-u)/2 < x$. Taking $\varepsilon = (x-u)/2$ in the hypothesis, we see that $x \notin S$. Thus u is an upper bound for S. The other part of the proof follows from the comment above. ∎

EXERCISES 5.1

1. Show that if a set has one upper bound, it has infinitely many.

2. Show that a set can have only one supremum and one infimum.

3. Show that if a set contains one of its upper (respectively, lower)

bounds, then that bound is the supremum (respectively, infimum) of the set.

4. (a) If $S \neq \emptyset$, show that $\inf S \leq \sup S$.

   (b) What can be said about the set S if $\inf S = \sup S$?

   (c) What are the supremum and infimum of $\emptyset$?

5. If S is a nonempty set, u is an upper bound for S, and v is not an upper bound for S, show that there is an element $s \in S$ with $v < s \leq u$.

6. (a) If $S \subseteq T$, show that $\inf T \leq \inf S$ and $\sup S \leq \sup T$.

   (b) Show that it is possible for the set containment in (a) to be proper without either of the inequalities being strict.

7. Show that a finite set contains its supremum and infimum.

8. (a) If all the elements of a set S are positive, show that $\inf S \geq 0$.

   (b) Show that a finite set whose elements are all positive has a *positive* infimum (remember, 0 is not positive).

   (c) Show that (b) does not hold if the set is infinite.

   (d) Show that if a set of positive numbers has infimum 0, the set *must* be infinite.

9. Suppose that $\sup(A \cup B) = u$ and that there is an $\varepsilon > 0$ so that $a < u - \varepsilon$ for all $a \in A$. Show that $\sup(A \cup B) = \sup B$.

10. (a) Show that if u and S are as in Theorem 5.3 and $u = \sup S$, then the other two conditions in the theorem are met.

    (b) Draw a picture to illustrate the proof of Theorem 5.3.

11. (a) Let S and T be sets with the property that $s < t$ for each $s \in S$ and each $t \in T$. Show that $\sup S \leq \inf T$.

    (b) If S and T are as in (a), show that $\sup S = \inf T$ if and only if, for any $\varepsilon > 0$, there are elements $s \in S$ and $t \in T$ with $t - s < \varepsilon$. (Note that $t - s > 0$.)

    (c) Let S and T be sets with the property that, for each $s \in S$, there is a $t \in T$ with $s < t$ and for each $t \in T$ there is an $s \in S$ with $s < t$. Show that $\inf S \leq \inf T$ and $\sup S \leq \sup T$.

    (d) Can these be replaced with strict inequalities?

    (e) Give a condition that would guarantee strict inequality in (a).

12. Define a property LR (for "left ray") as follows: The set A has LR if $x \in A$ and $y < x$ imply $y \in A$.

(a) Is a set having LR necessarily nonempty?

(b) If A has LR, $x \in A$ and $z \notin A$, then $z > x$.

(c) Does the whole number line have LR?

(d) If A and B have LR, either $A \subseteq B$ or $B \subseteq A$.

(e) Show that if $A \neq \emptyset$, A has LR, and $c = \sup A$, then either

$$A = \{x : x < c\} \quad \text{or} \quad A = \{x : x \leq c\}.$$

13. Discuss the relationship between our inability to complete Example 5.1.3 and our inability to show that the natural numbers are bounded above.

5.2 THE LEAST UPPER BOUND AXIOM

In Example 5.1.4 we saw a set of rational numbers that is bounded above but has no supremum. This is inconvenient. In the following axiom, we assert that it can't happen for a set of real numbers.

THE LEAST UPPER BOUND AXIOM: *Every nonempty subset of the real numbers that is bounded above has a least upper bound that is a real number.*

This is not true if "real" is replaced by "rational," as Example 5.1.4 demonstrates. *We have found what we were looking for*: a property that distinguishes the real numbers from the rational numbers. While it must remain an axiom for most of the book, the Least Upper Bound axiom will become a theorem in Chapter 22. We will say that an ordered field in which the Least Upper Bound axiom holds "has the Least Upper Bound property." Our first use of the Least Upper Bound axiom will be to complete Example 5.1.4.

THEOREM 5.4: (a) *There is no rational number whose square is 2.* (b) *Any ordered field having the Least Upper Bound property has a positive element whose square is 2.*

PROOF: (a) Suppose $(p/q)^2 = 2$, where p and q are integers. Then $p^2 = 2q^2$. Now p and q can be factored into prime numbers. The factor 2 may or may not appear in the prime factorization of p, but it must appear an *even* number of times (possibly none) in the factorization of p^2. Likewise, 2 must appear an even number of times in the factorization of q^2, and so 2 appears an *odd* number of times in the factorization of $2q^2$. The number of factors of 2 in a natural number can't be both even and odd, and so the assumption that $(p/q)^2 = 2$ has led to a contradiction.

(b) This is more complicated. Let $S = \{x \in \mathbf{F} : x > 0 \text{ and } x^2 < 2\}$. We show first that S has a supremum. Since $1 \in S$, we have $S \neq \emptyset$. If $x > 2$, then $x^2 > 4$ (by Theorem 4.12). Such a number is not an element of S. Thus 2 is an upper bound for S, and S is bounded above. Since $\mathbf{F}$ has the Least Upper Bound property, S has a supremum. The next step is to show that $(\sup S)^2 \geq 2$. Our proof will go like this: We will show that, for any $y \in \mathbf{F}$ with $y^2 < 2$, there is a positive element of $\mathbf{F}$, say z, with $(y+z)^2 < 2$. This says $y + z \in S$, and since $y < y + z$, we see that such a y can't be an upper bound for S. It follows that $y \neq \sup S$ and consequently that $(\sup S)^2 \not< 2$. A similar argument (which you will provide in Exercise 5.2.2) shows that $(\sup S)^2 \not> 2$. Combining the two inequalities, we have $(\sup S)^2 = 2$.

Now we will do the work. We indulge in a little "backward" thinking: We want $(y+z)^2 = y^2+2yz+z^2 < 2$. This is the same as $z(2y+z) < 2-y^2$, and we know that $2 - y^2$ is positive. How big is $z(2y + z)$? Since $y \in S$, we have $y < 2$. We must also have $y + z < 2$, and since $y > 0$, we may assume $z < 2$ (remember, we're working backward). Then $2y + z < 6$, and $z(2y + z) < 6z$. If we can make $6z < 2 - y^2$, we will be done. We can do this by letting $z = (2 - y^2)/7$. (The calculation should now be rewritten in the proper order.) ∎

The set $\{r \in \mathbf{Q} : r > 0 \text{ and } r^2 < 2\}$, thought of as a subset of $\mathbf{Q}$, is bounded above but has no supremum. When thought of as a subset of $\mathbf{R}$, though, the supremum of this set is a positive real number whose square is 2, which we can call $\sqrt{2}$.

EXERCISES 5.2

1. Show that the natural numbers have the Least Upper Bound property.

2. Complete the proof of Theorem 5.4.

3. (a) If S and T are bounded sets, show that $S \cap T$ and $S \cup T$ are bounded.

   (b) If S and T are as in (a), show that $\sup(S \cup T) = \max\{\sup S, \sup T\}$ (be sure to justify use of "max").

   (c) Is it true that $\sup(S \cap T) = \min\{\sup S, \sup T\}$?

   (d) Give a condition under which the equality in (c) would be true.

   (e) Let $\{S_\alpha : \alpha \in \mathcal{A}\}$ be a collection of bounded sets (where $\mathcal{A}$ is *finite*). Show that $\bigcup_{\alpha \in \mathcal{A}} S_\alpha$ is bounded.

   (f) Let $\{S_\alpha : \alpha \in \mathcal{A}\}$ be a collection of bounded sets (where $\mathcal{A}$ is *infinite*). Is $\bigcup_{\alpha \in \mathcal{A}} S_\alpha$ necessarily bounded?

4. (a) Does the collection of sets with the ordering given in Example 4.1.3 have the Least Upper Bound property?

   (b) What if the underlying set (the one whose subsets are ordered in this way) is infinite?

5. There is an element whose square is 2 in $\mathbf{R}$ but not in $\mathbf{Q}$. In $\mathbf{Z}_3$ we have $0^2 = 0$ and $1^2 = 2^2 = 1$, so there is no element in $\mathbf{Z}_3$ whose square is 2. Are there any values of p for which $\mathbf{Z}_p$ has an element whose square is 2? Can you characterize those numbers p for which $\mathbf{Z}_p$ has an element whose square is 2?

6. For any set S, let $-S = \{x : x = -s \text{ for some } s \in S\}$. If S is bounded below, show that $-S$ is bounded above and that $\sup(-S) = -\inf S$ (this allows us to use theorems about suprema to say things about infima).

7. Here are two more ways to prove that $\sqrt{2}$ is irrational:

   (a) (i) Show that if n is a natural number and n^2 is even, then n is even. (ii) Assume that $p^2 = 2q^2$ and that p and q have no common factors. Derive a contradiction.

   (b) (i) Show that if x is a positive number with $x^2 = 2$, then $1 < x < 2$. (ii) Let $S = \{n \in \mathbf{N} : n\sqrt{2} \in \mathbf{N}\}$. If $\sqrt{2}$ is rational, we have $S \neq \emptyset$. Let q be the least element of S. Examine $q\sqrt{2} - q$ and derive a contradiction.

   (c) One's preference of a proof of the irrationality of $\sqrt{2}$ depends in part on which number-theoretic results[1] one is willing to consider most evident. Review the three proofs of the irrationality of $\sqrt{2}$ and pick out the number-theoretic results that are needed to make each of them work.

8. The following "proof" contains at least two serious errors. Draw a picture to illustrate what the author of this "proof" *thought* they were doing. Find the errors and explain why they are errors. Finally, give an example to show that the result is false.

   "THEOREM": Every nonempty set is a neighborhood of at least one of its points.

   "PROOF": Let A be a nonempty set. Let $u = \sup A$. Since $u = \sup A$, there is a number $\varepsilon > 0$ so that $(u - \varepsilon, u) \subseteq A$. Let

[1] "Number-theoretic results" include, among other things, statements concerning the arithmetic of the natural numbers, which natural numbers divide evenly into which others, and the ways numbers can be factored.

$x = u - \varepsilon/2$. Now the interval $(x - \varepsilon/4, x + \varepsilon/4)$ is contained in the interval $(u - \varepsilon, u)$, and so $(x - \varepsilon/4, x + \varepsilon/4)$ is contained in A. Thus A is a neighborhood of x.

9. (a) If S is a nonempty bounded set, show that $S \subseteq [\inf S, \sup S]$.

   (b) If S is as in (a) and I is a closed interval with $S \subseteq I$, show that $[\inf S, \sup S] \subseteq I$.

   (c) If S is as in (a), show that $[\inf S, \sup S] = \bigcap I$, where the intersection is taken over all closed intervals containing S.

   (d) What is $\bigcap I$, where the intersection is taken over all *open* intervals containing S?

10. (a) Let S be a nonempty set that is bounded above, and let
$$T = \{x : x \text{ is an upper bound for } S\}.$$
Show that T is nonempty and bounded below and that $\sup S = \inf T$.

    (b) Let S be a nonempty set that is bounded below, and let
$$T = \{x : x \text{ is a lower bound for } S\}.$$
Show that T is nonempty and bounded above and that $\sup T = \inf S$.

    (c) Use (b) to establish a Greatest Lower Bound axiom. (Since you will be proving it, this will be a Greatest Lower Bound *theorem*.)

11. (a) Modify the proof of Theorem 5.4 to show that
    (i) there is no rational number whose square is 3, and
    (ii) in any ordered field in which the Least Upper Bound axiom holds, there is a positive element whose square is 3.

    (b) Where does the proof of the first part of Theorem 5.4 fail if one tries to use the same method to show that there is no rational number whose square is 4?

12. (a) Define the sum of two sets A and B to be
$$A + B = \{z : z = x + y \text{ for some } x \in A \text{ and } y \in B\}.$$
If A and B are bounded above, show that $A + B$ is bounded above and that $\sup(A + B) = \sup A + \sup B$.

    (b) Define a "multiple" of a set to be
$$kA = \{z : z = kx \text{ for some } x \in A\}.$$
If $k > 0$ and A is bounded above, show that kA is bounded above and $\sup kA = k(\sup A)$. What happens if $k < 0$?

    (c) We have definitions of "addition" and "scalar multiplication" for sets. Is the collection of subsets of the real line a vector space under this structure?

(d) Along these same lines, we could define the "product" of two sets to be $AB = \{z : z = xy \text{ for some } x \in A \text{ and } y \in B\}$ Show that it is not true in general that $\sup AB = (\sup A)(\sup B)$. Is this *ever* the case?

13. Let $[\![x]\!]$ denote the greatest integer less than or equal to x (so that, for instance, $[\![\pi]\!] = 3, [\![-5.79]\!] = -6$, and $[\![4]\!] = 4$).

(a) Show that a number x is an integer if and only if $[\![x]\!] = x$.

(b) Show that x is rational if and only if there is a natural number n so that $[\![nx]\!] = nx$.

(c) Recall from calculus that $e = \sum_{k=0}^{\infty} \frac{1}{k!}$. For any natural number n, show that $[\![n!e]\!] = n! \sum_{k=0}^{n} \frac{1}{k!}$.

(d) Observe that $n! \sum_{k=0}^{n} \frac{1}{k!} < n!e$ for all n. Show that e is irrational.

14. (a) Examine the numbers $\sqrt{2}^{\sqrt{2}}$ and $\left[\sqrt{2}^{\sqrt{2}}\right]^{\sqrt{2}}$ to show that it is possible for an irrational number raised to an irrational power to be rational. (Is the first one rational? If not, is the second one rational?)

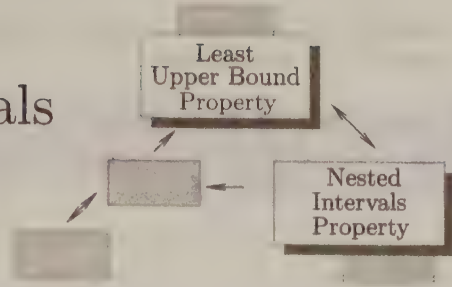
(b) Consider other combinations of bases and exponents. For instance, is it possible for an irrational number raised to a rational power to be rational? Consider what might happen if the base (or power) is an integer.

(c) Think about whether you have *really* proved the result in (a). It seems that one of the two numbers given must be an example of the phenomenon, but which one is it? Have you really given an example? (This is an exercise in intuitionist thinking. See Exercise 1.3.1.)

(d) Upon begin posed the question in (a)—Can an irrational number raised to an irrational power be rational?—many people respond "Just look at $e^{\ln 2}$." Is this answer any better (or worse) than the one we've given here?

Chapter 6

Nested Intervals



6.1 THE INTEGER PART OF A NUMBER—THE ARCHIMEDEAN PROPERTY

In this chapter we will finally establish the association between elements of an ordered field and decimal expansions. Two theorems (both of which appear at first to be about other things altogether!) will lead us to this reassuring result. The first of these theorems lets us find the integer part of an element of our field. For the moment, we will consider only positive elements of the ordered field. If $x \in \mathbf{P}$ and n is a natural number with $x \in (n-1, n]$, we call $n-1$ the integer part[1] of x. We will show that such a natural number exists by attacking a similar but smaller problem. Instead of finding what seems to be the *smallest* natural number larger than x, we will show first that x is smaller than *some* natural number. This is not as obvious as it might seem.

Recall the field of formal rational functions in Chapter 3. We may make this an ordered field by saying that $p(x)/q(x)$ is positive if the coefficients of the highest-order terms of $p(x)$ and $q(x)$ have the same sign. Now $x/1$ is a formal rational function, as is $n/1$ (which corresponds to the natural number n in this field). Note that $(x/1)-(n/1) = (x-n)/1$ is positive no matter what n is. In other words, $x/1$ is *larger than all the natural numbers* in this field. Recall that there is no such element in the rational numbers (Exercise 4.6.8). While the structure of the field of formal rational functions remains somewhat of a mystery, we now know that this field is definitely different from the rational numbers.

Theorem 6.1 tells us that, in an ordered field with the Least Upper Bound property, there is no element larger than all the natural numbers. The conclusion of Theorem 6.1 is called the **Archimedean property**.

[1] In keeping with our earlier stipulation that there should be no terminating decimals, the integer part of the natural number m is $m-1$ (so, for instance, the decimal expansion of the natural number 23 will turn out to be 22.999...).

A field for which it holds is said to "have the Archimedean property" or simply to "be Archimedean."

THEOREM 6.1: *Let $(\mathbf{F}, \mathbf{P})$ be an ordered field having the Least Upper Bound property, and let x be any element of $\mathbf{F}$. Then there is a natural number n_x with $n_x > x$.*

PROOF: This statement is the same as saying $\mathbf{N} \subseteq \mathbf{F}$ is not bounded above. Suppose $\mathbf{N}$ is bounded above. Then, by the Least Upper Bound property,[2] $\mathbf{N}$ must have a supremum, call it u. Taking $\varepsilon = 1$ in Theorem 5.3, there is a natural number n_0 with $n_0 > u - 1$. Now $n_0 + 1$ is also a natural number, and $u < n_0 + 1$. This contradicts the assertion that $u = \sup \mathbf{N}$ (because u is not an upper bound for $\mathbf{N}$) and thus contradicts the assumption that $\mathbf{N}$ is bounded above. ∎

The Archimedean property has several important corollaries:

COROLLARY 6.2: *If $(\mathbf{F}, \mathbf{P})$ is an ordered field having the Least Upper Bound property, then*

(a) If $x \in \mathbf{P}$, there is a natural number n_x with $n_x - 1 < x \leq n_x$ (this allows us to define the integer part of a positive number).

(b) If $x \in \mathbf{P}$, there is a natural number n_x with $1/n_x < x$ (this allows us to complete Example 5.1.3).

(c) For any $x, y \in \mathbf{P}$, there is a natural number n so that $nx > y$.

(d) Every nonempty open interval in $\mathbf{F}$ contains both a rational element of $\mathbf{F}$ and an irrational element of $\mathbf{F}$.

PROOF: The proofs of parts (b) and (c) are left as Exercise 6.1.2. (a) Let $x \in \mathbf{P}$. By the Archimedean property, $B = \{k \in \mathbf{N} : k \geq x\} \neq \emptyset$. According to the well-ordering property, B has a least element. Call it n_x. Then $n_x \geq x$ because $n_x \in B$, and $n_x - 1 < x$ because $n_x - 1 \notin B$.

(d) Let the interval be (a, b). Since $b - a > 0$, part (c) of the Corollary says there is a natural number n with $n(b - a) > 1$. By Exercise 4.9.3, there is a natural number m in (na, nb). Then $na < m < nb$, and, since $n > 0$, we may divide to obtain $a < m/n < b$. This is the rational element we were after. Now let p be a rational element with $a < p < b$, and r a rational element with $a < p < r < b$. Then $(1/\sqrt{2})p + (1 - 1/\sqrt{2})r$ is an

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[2] Notice that the assumption that $\mathbf{N}$ is bounded above gives us more information than we had before, since it allows us to use the Least Upper Bound property. This is a good proof to do by contradiction.

irrational element of the type we were looking for (you will check this in Exercise 6.1.3). ∎

A subset of the real numbers is called **dense in the real line** (or just **dense**) if its intersection with any nonempty open interval is nonempty. Corollary 6.2.d says that the rational numbers and the irrational numbers are both dense in the real line. It is often called the **Density theorem** for this reason. We may sharpen the second part of it in this way:

THEOREM 6.3: *If* $\mathbf{F}$ *is an Archimedean ordered field having an irrational element* z*, and* $a < b$*, there is a rational element* r *with* $rz \in (a, b)$.

PROOF: Left as Exercise 6.1.5. ∎

Theorem 6.3 is quite relevant to us, since we are only certain that one number ($\sqrt{2}$) is irrational. We will find that having a specific example of an object is often more useful than knowing that lots of them exist. On the other hand, since, as we will see shortly, there are uncountably many irrational numbers, there must be at least one of them. There must be irrational numbers because there aren't enough rational numbers to account for all the real numbers. This counting technique to prove that something exists is very handy. We used it in Exercise 2.2.5 to show that there must be transcendental numbers, thereby saving us the trouble of actually having to find one.

EXERCISES 6.1

1. (a) Verify that the formal rational functions are an ordered field with the positive set given.

   (b) Show that $(x/1) - (n/1) \in \mathbf{P}$ for all n.

2. (a) Prove Corollaries 6.2.b and 6.2.c.

   (b) Modify Corollary 6.2.a to include negative numbers.

   (c) If $a < b$, show that $(a, b) \cap \mathbf{Q}$ and $(a, b) \cap \mathbf{R}\backslash\mathbf{Q}$ are both *infinite*. (Corollary 6.2.d says they are nonempty.)

   (d) If D is *any* dense subset of the real numbers and $a < b$, show that $D \cap (a, b)$ is infinite. (Again, the definition of dense only stipulates that this intersection is nonempty.)

   (e) Show that no finite set can be dense in the real line.

3. (a) If $0 < t < 1$ and $a < b$, show that $a < ta + (1 - t)b < b$.

   (b) Complete the proof of Corollary 6.2.d.

4. Use Exercise 4.10.3 to prove the Density Theorem (as it relates to irrational numbers).

5. Prove Theorem 6.3.

6. (a) Suppose x_k is a real number for $k = 1, 2, \ldots$, and that there is a positive number ε so that $x_k > \varepsilon$ for each k. If B is any real number, show that there is a natural number n so that $x_1 + x_2 + \cdots + x_n > B$.

   (b) Show that this need not be the case if we assume only that $x_n > 0$ for all n.

7. Use Bernoulli's inequality (Exercise 4.5.19) to show that if $x > 1$, the set of numbers $\{x^n\}$ is unbounded.

8. (a) Show in detail that $\inf\left\{\frac{1}{2^n}\right\} = 0$.

   (b) Why is this exercise here rather than in Chapter 5?

9. (a) If T is a linearly ordered set and $S \subseteq T$, we say that S and T are **coterminal** if for each $t \in T$ there is an $s \in S$ with $s > t$, and vice versa. If T and S are coterminal ordered fields, show that T is Archimedean if and only if S is Archimedean.

   (b) Could we use this result, along with Exercise 4.6.8, to show that the real number line is Archimedean?.

10. Suppose S and U are sets with the following two properties:

    (i) Each element of U is an upper bound for S.

    (ii) For any $n \in \mathbf{N}$, there are $s \in S$ and $u \in U$ with $u - s < 1/n$.

    (a) Show that each element of S is a lower bound for U.

    (b) Show that $\sup S = \inf U$.

    (c) If u is an upper bound for a set X with the property that for any $n \in \mathbf{N}$ there is an $x \in X$ with $u - x < 1/n$, then $u = \sup X$.

    (d) Show that u is an upper bound for a set X if and only if $u + 1/n$ is an upper bound for X for all n.

    (e) Combine (c) and (d) to show: $u = \sup X$ if and only if, for each natural number $n, u + 1/n$ is an upper bound for X and $u - 1/n$ is not.

    (f) This question is about suprema. Why is it in this chapter instead of in the previous one?

11. (a) Show that an element of the positive set in the field of formal rational functions, considered now as a genuine function, does *not*

necessarily take on only positive values. Can we say anything about the values of a "positive" formal rational function?

(b) Define a subset **P** of the field of formal rational functions by

$$\mathbf{P} = \{f : f(x) > 0, \forall x \in \mathbf{R}\}$$

Is this a positive set?

12. (a) If D is dense in the the real line and $D \subseteq S$, show that S is dense in the real line.

 (b) Show that if S is dense in the real line and a finite number of points are removed from S, the resulting set is also dense in the real line.

 (c) Does (b) necessarily remain true if the set that is removed is infinite?

 (d) Show that every dense set has a proper subset that is also dense.

13. The **dyadic rationals** are rational numbers of the form $n/2^m$ for some integer n and some natural number m.

 (a) Show that not every rational number is a dyadic rational.

 (b) Show that the set of dyadic rationals is countable.

 (c) Show that the set of dyadic rationals is dense in the real line.

 (d) Show that all of this is true if the denominators of the fractions are powers of any natural number greater than 1 (of course, such things aren't the dyadic rationals anymore).

14. (a) Show that the set of natural numbers does not have a supremum in any ordered field (Archimedean or not).

 (b) Let **F** be a non-Archimedean ordered field, and let

$$U = \{x : x \text{ is an upper bound for } \mathbf{N}\}$$

 Show that (i) $U \neq \emptyset$, and (ii) U is bounded below but has no infimum.

 (c) Doesn't part (a) say that *every* ordered field is Archimedean?

6.2 NESTS

We will now find the fractional part of an element of our field. As before, the beginning of this journey may not immediately seem related to its end.

DEFINITION 6.4: A collection of sets, $S_1, S_2, \ldots$, is a **nest** if $S_n \supseteq S_{n+1}$ for $n = 1, 2, \ldots$.

We are interested only in nests of intervals of real or rational numbers, say $\{[a_n, b_n]\}$, which we may visualize like this:

It need not happen (as it does in this diagram) that $a_n < a_{n+1}$ and $b_n > b_{n+1}$ for all n. Consecutive endpoints (or consecutive sets for that matter) might be the same.

THEOREM 6.5: (The Nested Intervals Property) *If* $\mathbf{F}$ *is an ordered field having the Least Upper Bound property, then*

(a) *Any nest* $\{I_n\}$ *of nonempty, closed, bounded intervals has a nonempty intersection (that is,* $\bigcap_n I_n \neq \emptyset$*).*

(b) *If the infimum of the lengths of the intervals* I_n *is 0, there is an element of* $\mathbf{F}$*, say* x*, so that* $\bigcap_n I_n = \{x\}$.

PROOF: (a) Let $I_n = [a_n, b_n]$. Since $I_n \supseteq I_{n+1}$, we have $a_1 \leq a_2 \leq \ldots$ and $b_1 \geq b_2 \geq \ldots$ (by Exercise 4.9.4). We also have $a_1 \leq b_1, a_2 \leq b_2, \ldots$. We begin by showing that $a_i \leq b_j$ for every i and j (note that this is *not* part of the "given"). There are three cases to consider, depending on whether $i = j$, $i < j$, or $i > j$. If $i = j$, it *is* given that $a_i \leq b_j$. If $i < j$, note that $a_i \leq a_j \leq b_j$. Finally, if $i > j$, then $a_i \leq b_i \leq b_j$ (it is easy to see what has happened here by looking at i and j equal to 2 or 3 in the diagram above).

Let $A = \{a_1, a_2, \ldots\}$. Then b_1 is an upper bound for A (as is each b_n). By the Least Upper Bound property, A has a supremum. Call it x. By the definition of supremum, $x \geq a_n$ for each n and, since each b_n is an upper bound for $A, x \leq b_n$ for each n (be sure you see why this is so). So $a_n \leq x \leq b_n$. That is, $x \in I_n$ for each n. It follows that $x \in \bigcap_n I_n$, and so this intersection is not empty.

(b) Suppose x and y are elements of $\bigcap_n I_n$. Then x and y are in I_n for each n, and so $|x - y| \leq b_n - a_n$ for each n (Theorem 4.19). Since $\inf\{b_n - a_n\} = 0$, and $|x - y| \geq 0$, it can only be that $|x - y| = 0$, that is, $x = y$. ∎

EXERCISES 6.2

1. In the notation of the proof of Theorem 6.5, show that
$$\bigcap_n [a_n, b_n] = [\sup\{a_n\}, \inf\{b_n\}].$$

2. Give an example of a nest of *bounded* intervals whose intersection is empty and of a nest of *closed* intervals whose intersection is empty.

3. Can the Nested Intervals property be weakened to allow intervals that are not closed? For instance, does the result remain true if the intervals are required to be of the form $[a, b)$? How about $(a, b]$? What if some of the intervals are of the form $[a, b)$ and some are of the form $(a, b]$?

4. (a) In the field of formal rational functions, construct a nest of closed, bounded intervals whose intersection is empty. (That is, show that the Nested Intervals property fails in this field.)

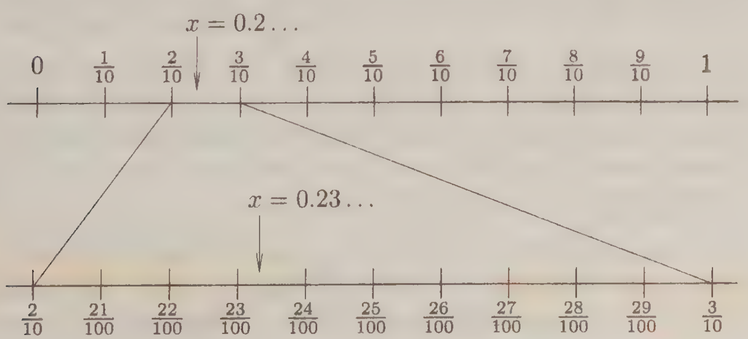
   (b) Discuss whether the Nested Intervals property must fail in *any* non-Archimedean ordered field.

6.3 THE FRACTIONAL PART OF A NUMBER—DECIMAL EXPANSIONS

We can now complete the association between elements of a complete ordered field and decimal expansions. Having done this, we may feel secure in referring to this field as the real numbers. If k is the integer part of the positive number x (provided by Corollary 6.2.a), then $x - k \in (0, 1]$. We will devise a procedure for associating each element of the interval $(0, 1]$ with a decimal expansion, and vice versa.

THEOREM 6.6: *In an Archimedean ordered field in which the Nested Intervals property holds, there is a one-to-one correspondence between the interval $I = (0, 1]$ and the nonterminating decimal expansions of the form $0.d_1d_2d_3\ldots$.*

PROOF: Let $x \in I$. We will describe a decimal expansion of the proper form (if you fuss over all the details of this argument, you will see a good illustration that clarity and precision do not always go hand in hand).



Divide I into 10 disjoint subintervals of the form $(a, b]$, each of length 1/10. Since these intervals are disjoint and their union is I, one of them (and only one of them) must contain x. If x is in the kth interval from the left, set $d_1 = k - 1$ For instance, $d_1 = 2$ if x is in $(2/10, 3/10]$. Now divide the interval just selected into 10 disjoint subintervals of length 1/100 and choose d_2 in the same way. This is illustrated in the diagram above. Continue in this way, dividing the current interval into 10 parts and assigning the next decimal place. We have associated x with a decimal expansion. That this association is one-to-one is seen as in the proof of Theorem 6.5.b.

We've done only half the proof (and we haven't used the Nested Intervals property yet!) We have to show that this association is onto. We now show that every decimal expansion corresponds to an element of I (that is, that this correspondence between numbers and decimals is onto). This is quite simple, since the process we've just described can be reversed. Just use the digits to select the intervals instead of the other way around. We must replace half-open intervals with the associated closed ones in order to use the Nested Intervals property. Observe that the infimum of the lengths of the intervals in the resulting nest is 0. By the Nested Intervals property, there is a number x that is the only element of the intersection of this nest. This is the number to which the decimal corresponds. ∎

If k is the integer part of $x > 0$, and $0.d_1d_2d_3\ldots$, is the decimal expansion of $x - k$, found as above, then the decimal expansion of x is $k.d_1d_2d_3\ldots$.

COROLLARY 6.7: *There are uncountably many real numbers. There are uncountably many irrational numbers.*

PROOF: The first statement follows from Exercise 2.3.6, Cantor diagonalization, and Theorem 6.6. Note that $\mathbf{R} = \mathbf{Q} \cup \mathbf{R} \backslash \mathbf{Q}$. If the set of irrational numbers were countable, we would have written $\mathbf{R}$ as a union of two countable sets and $\mathbf{R}$ would be countable. ∎

Corollary 6.7 answers the Big Question to the extent that we now know that the real numbers and the rational numbers are indeed different. It might be a bit unsatisfying that this answer does not refer to any aspect of the sets other than cardinality. Here is an example that involves the order structure:

EXAMPLES 6.3: 1. By Theorem 5.4, we know there is an irrational number whose square is 2. By Theorem 6.6, this number has a decimal expansion (which we suspect begins $1.414\ldots$). Consider the intervals

$J_0 = [1, 2], J_1 = [1.4, 1.5], J_2 = [1.41, 1.42], \ldots$. (If the truncation of the decimal expansion of $\sqrt{2}$ to n places is r_n, let $J_n = [r_n, r_n + 10^{-n}]$.) Note that $\sqrt{2} \in J_n$ for all n (this is how the decimal places were chosen). Let $I_n = J_n \cap \mathbf{Q}$. Since the endpoints of each J_n are rational, each I_n contains its endpoints. Thus each I_n is a closed, bounded interval of rational numbers. But $\bigcap_n I_n = (\bigcap_n J_n) \cap \mathbf{Q} = \{\sqrt{2}\} \cap \mathbf{Q} = \emptyset$. Thus $\{I_n\}$ is a nest of closed, bounded intervals of rational numbers whose intersection is empty. We see that $\mathbf{Q}$ doesn't have the Nested Intervals property.

EXERCISES 6.3

1. (a) By splitting the intervals in halves instead of tenths, modify Theorem 6.6 to show that every element of a complete ordered field has a **binary** (base 2) expansion.

   (b) Show that every element of a complete ordered field has a **ternary** (base 3) expansion.

   (c) Would the process described in Theorem 6.6 still work if one used different numbers of intervals at each stage? Is there any reason to do this?

2. (a) What decimal expansion does Theorem 6.6 assign the number 1?

   (b) Show that Theorem 6.6 doesn't assign any number a terminating decimal expansion.

   (c) Describe the numbers that Theorem 6.6 assigns decimal expansions ending in an infinite string of 9s.

   (d) Show that the Theorem 6.6, as modified in Exercise 1, doesn't assign any number a terminating expansion.

   (e) Describe the numbers that the procedure in Exercise 1 assigns binary expansions ending in an infinite string of 1s. Describe the numbers that the procedure in Exercise 1 assigns ternary expansions ending in an infinite string of 2s.

   (f) Suppose we begin by dividing the intervals into b equal parts. Repeat this exercise for such a procedure. (Note then that *every element of a complete ordered field has an expansion in any number base.*)

3. Modify the discussion in the chapter to show that negative numbers have decimal expansions.

4. (a) Suppose we've selected a finite portion of the decimal expansion for x as in Theorem 6.6: $0.d_1d_2 \ldots d_{n-1}d_n$. Show that

$$\frac{d_1}{10} + \frac{d_2}{100} + \cdots + \frac{d_n}{10^n} < x \leq \frac{d_1}{10} + \frac{d_2}{100} + \cdots + \frac{d_n + 1}{10^n}$$

(b) Show that the decimal expansion of $\sqrt{2}$ begins 1.414....

5. How is the Archimedean property used in the proof of Theorem 6.6?

6. The intervals J_n in Example 6.3.1 were chosen to make $\bigcap_n J_n = \{\sqrt{2}\}$. Why couldn't we have used the intervals $[\sqrt{2} - 1/n, \sqrt{2} + 1/n]$ and get the same result?

7. (a) Show that any repeating decimal represents a rational number and that every rational number is represented by a repeating decimal.

   (b) Show that there are countably many repeating decimals.

   (c) Show that there are uncountably many nonrepeating decimals.

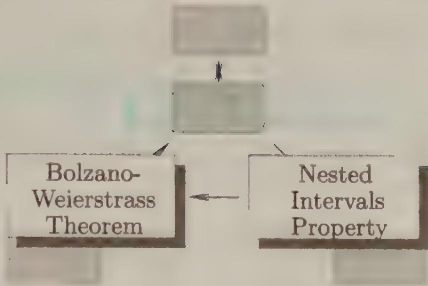
   (d) Why does this exercise appear here rather than in Chapter 2?

8. Comment on the following as a definition of addition for real numbers: Suppose the integer parts of x and y are m and n and the fractional parts of x and y are associated by Theorem 6.6 with nests $\{[a_n, b_n]\}$ and $\{[c_n, d_n]\}$. Note that a_n, b_n, c_n, and d_n are rational (so we know how to add them). *Let z be the unique element of $\bigcap_n [a_n + c_n, b_n + d_n]$.* If $z < 1$, let $x + y$ have integer part $m + n$ and fractional part z. If $z > 1$, let $x + y$ have integer part $m + n + 1$ and fractional part the same as the fractional part of z. If $z = 1$, let $x + y = m + n + 1$. (You should begin by examining closely the sentence in italics.)

9. Prove Theorem 6.1 again by showing that no element of an ordered field can possibly satisfy the conditions of Theorem 5.3 and consequently that no element of such a field can be the supremum of $\mathbf{N}$.

Chapter 7

Cluster Points



7.1 POINTS AND SETS

A real number and a set of real numbers might be related in several ways. The point might be an element of the set or it might not, but this is just the simplest case. Our work has shown us other possibilities:

(1) The set might be a neighborhood of the point.
(2) The point might be an upper or lower bound for the set.
(3) The point might be the supremum or infimum of the set.
(4) The point might be related to the *complement* of the set in one of these ways.

These statements are not unrelated: The supremum of a set is an upper bound of the set; a set contains any point of which it is a neighborhood; and so on. On the other hand, a set *can't* be a neighborhood of its supremum (be sure you see why this is so). Something that can't happen is often as interesting as something that can (or must). We will look at this list from two points of view and from different beginnings arrive at much the same end.

EXERCISES 7.1

1. Show that a set can't be a neighborhood of its supremum.

2. (a) Show that a closed, bounded interval contains is supremum and infimum and that an open interval contains neither.

   (b) Give examples of (i) a set that is not a closed, bounded interval, but nevertheless contains its supremum and infimum, and (ii) a set that is not an open interval but nevertheless does not contain it supremum or infimum.

7.2 ONE POINT OF VIEW

Loosely speaking, a set is a neighborhood of a point if the set contains everything "near" the point. Can we weaken this requirement? Take a set S and a point s and consider the following statements, the first being the definition of neighborhood:

(1) There is an $\varepsilon > 0$ so that S contains *all* points of $(s - \varepsilon, s + \varepsilon)$.

(2) There is an $\varepsilon > 0$ so that S contains *all but finitely many* points of $(s - \varepsilon, s + \varepsilon)$.

(3) There is an $\varepsilon > 0$ so that S contains *infinitely many* points of $(s - \varepsilon, s + \varepsilon)$.

(4) There is an $\varepsilon > 0$ so that S contains *a point other than s* of $(s - \varepsilon, s + \varepsilon)$.

(5) For every $\varepsilon > 0$, S contains *infinitely many* points of $(s-\varepsilon, s+\varepsilon)$.

(6) For every $\varepsilon > 0$, S contains *a point other than s* of $(s - \varepsilon, s + \varepsilon)$.

Each of statements (2), (3), and (4) seems to be a weakening of the one before it. (When we say statement (2) is weaker than statement (1), we mean there might be more points and sets satisfying the former than the latter.) Statements (5) and (6) are altered in a different way. Though the second statement seems weaker than the first, in a sense this is not really so:

THEOREM 7.1: (a) *Statement* (1) *implies statement* (2);
(b) *If $s \in S$, then statement* (1) *and statement* (2) *are equivalent.*

PROOF: (a) Is clear, since $\emptyset$ is finite;
(b) If statement (2) holds, then $\{|x - s| : x \in (s - \varepsilon, s + \varepsilon)\backslash S\}$ is a finite set of positive numbers, and so it has a positive infimum (Exercise 5.1.8). If ε_1 is this infimum, then $(s - \varepsilon_1, s + \varepsilon_1) \subseteq S$ and statement (1) holds. ∎

If s is one of the "missing points" in statement (2), S is called a **deleted neighborhood** of s. These are of considerable importance in calculus but of little interest to us now (statements like "$f(x)$ does so and so if $0 < |x - a| < \delta$" are references to deleted neighborhoods). It is left to the reader to decide that the situations described in statements (3) and (4) are so simple as to be uninteresting.

THEOREM 7.2: *Statements* (5) *and* (6) *above are equivalent.*

PROOF: Clearly statement (5) implies statement (6) since an infinite

set has at least two elements (one of which might be s). Let $\varepsilon > 0$ be given, and suppose $S \cap (s - \varepsilon, s + \varepsilon)$ contains a point other than s. Call this point s_1. Now let $\varepsilon_1 = |s - s_1| > 0$. There is a point of S other than s in $(s - \varepsilon_1, s + \varepsilon_1)$; call it s_2. Note that $s_2 \neq s_1$. Continuing in this way, we find $s_1, s_2, s_3, \ldots$, infinitely many points of $S \cap (s - \varepsilon, s + \varepsilon)$. ∎

EXERCISES 7.2

1. Draw a picture to illustrate the proof of Theorem 7.1.b.

2. Explain the comment about statements (3) and (4).

7.3 ANOTHER POINT OF VIEW

Consider the statement "S is *not* a neighborhood of s." Then it is *not* the case that $\exists \varepsilon > 0 \ni ((s - \varepsilon, s + \varepsilon) \subseteq S)$. That is, $\forall \varepsilon > 0((s - \varepsilon, s + \varepsilon) \not\subseteq S)$ or $\forall \varepsilon > 0((s - \varepsilon, s + \varepsilon) \cap C(S) \neq \emptyset)$, which is very much like statement (6) applied to the set $C(S)$ (how is it different?). A point seems to have the property we're talking about (whatever that might be) as it relates to $C(S)$ if S is *not* a neighborhood of it.

7.4 CLUSTER POINTS

We will not have much occasion to use the latter characterization. We have mentioned it only to show that the idea can arise in more than one way. Statements (5) and (6) are equivalent, and so we may take either of them as our definition. The fifth is a bit more suggestive:

DEFINITION 7.3: The point s is a **cluster point** of the set S if, for every $\varepsilon > 0$, the set $(s - \varepsilon, s + \varepsilon) \cap S$ is infinite.[1]

This is the same as requiring the intersection of every *neighborhood* of s with S to be infinite. Theorem 7.2 says that a point s is a cluster point of a set S if and only if, for every $\varepsilon > 0$, $S \cap (s - \varepsilon, s + \varepsilon) \setminus \{s\} \neq \emptyset$. Knowing that a set is infinite would seem to be more useful than knowing just that it is not empty. Curiously, it is often easier to show that a set is infinite than to show that it is not empty. It is rare that the more useful bit of information is easier to come by.

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[1] Some authors use the phrase "accumulation point" or "limit point," while others use the same words with slight differences in meaning. One must be careful to check how a particular author uses these terms.

EXAMPLES 7.4: 1. A finite set has no cluster points since it is impossible for its intersection with any set to be infinite. This is not the only way a set can fail to have a cluster point. The set of natural numbers is infinite, yet it has no cluster point since $(x - \varepsilon, x + \varepsilon) \cap \mathbf{N}$ is finite for any x and any $\varepsilon > 0$.

2. $1/2$ is a cluster point of $(0, 1)$. If $\varepsilon \geq 1/2$, then $(1/2 - \varepsilon, 1/2 + \varepsilon) \cap (0, 1) = (0, 1)$. If $\varepsilon < 1/2$, then $(1/2 - \varepsilon, 1/2 + \varepsilon) \cap (0, 1) = (1/2 - \varepsilon, 1/2 + \varepsilon)$. In either case, the intersection is an open interval, which is an uncountable set by Exercise 2.1.2. Notice that 0 is also a cluster point of $(0, 1)$ since $(0, 1) \cap (0 - \varepsilon, 0 + \varepsilon) = (0, \min\{\varepsilon, 1\})$, which is an infinite set.

We can conclude from this example that if S is a neighborhood of s, then s is a cluster point of S. But this does not work the other way: 0 is a cluster point of $(0, 1)$ even though $(0, 1)$ is not a neighborhood of 0. It is not necessary for a set to be a neighborhood of a point for the point to be a cluster point of the set, in fact, *it is not necessary for a point to be an element of a set for the point to be a cluster point of the set.* Furthermore, in view of Example 1, a point that is an element of a set need not be a cluster point of the set.

3. 0 is a cluster point of $H = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. If $\varepsilon > 0$ is given, and n is such that $1/n < \varepsilon$ (by Corollary 6.2.b), then $\{\frac{1}{n}, \frac{1}{n+1}, \ldots\} \subseteq (0 - \varepsilon, 0 + \varepsilon) \cap H$, so this intersection is infinite.

EXERCISES 7.4

1. Show that s is a cluster point of S if and only if $S \cap U$ is infinite whenever U is a neighborhood of s.

2. If S is a set that is bounded above, show that $\sup S$ is either an element of S or is a cluster point of S.

3. We may say that x is a "right" cluster point of a set S if, for any $\varepsilon > 0$, $S \cap (x, x + \varepsilon) \neq \emptyset$, and similarly for "left" cluster points. We could also insist that these intersections be infinite.

   (a) Show that the two definitions of right cluster point suggested above are equivalent, and the two definition of left cluster point suggested above are equivalent.

   (b) Show that every right cluster point is a cluster point, but that not every cluster point is a right cluster point. Similarly for left cluster points.

(c) Examine the examples of cluster points in the chapter. Which are right cluster points and which are left cluster points?

(d) If x is a cluster point of S, must it be the case that x is either a right cluster point or a left cluster point of S?

7.5 DERIVED SETS

We'll take a detour at this point to identify the set of all cluster points of (0, 1). Every element of $(0,1)$ is a cluster point of it [since $(0,1)$ is a neighborhood of each of its points], and 0 and 1 are also cluster points. Are there any others? If $x < 0$, the interval $(x-1,0)$ is a neighborhood of x containing no element of $(0,1)$, and so x is not a cluster point of $(0,1)$. Similarly, if $x > 1$, the interval $(1,x+1)$ is a neighborhood of x containing no element of $(0,1)$. Thus the set of cluster points of $(0,1)$ is $[0,1]$. The set of cluster points of a set S is called the **derived set** of S and is denoted S' (read "S prime"). We see that a derived set might be larger than the original set. We will frame the rest of our examples in terms of derived sets.

EXAMPLES 7.5: 1. It is also the case that $[0,1]' = [0,1]$. Most closed intervals are equal to their derived sets. The only exceptions are those that consist of a single point. For instance, $[3,3]$ is a closed interval but has no cluster point (since it is finite). In any case, though, a closed interval *contains all of its cluster points*, and an open interval does not.

2. Consider $H = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ again. We have shown that $0 \in H'$. If $x < 0$, then $(x-1,0)$ is a neighborhood of x whose intersection with H is empty, so x is not a cluster point of H. If $x > 0$, then $(\frac{x}{2},\infty)$ is a neighborhood of x whose intersection with H is finite (be sure you see why). Thus 0 is the *only* cluster point of H; that is, $H' = \{0\}$. The derived set of an infinite set can be finite (or empty), and we see that a set can be disjoint from its derived set. Occasionally we will want to build sets with specified cluster points. This example gives us a hint how to do it.

3. $\mathbf{Q}' = \mathbf{R}$. By the Density theorem, if x is any real number and $\varepsilon > 0$, then $(x, x+\varepsilon) \cap \mathbf{Q} \neq \emptyset$, and so $(x-\varepsilon, x+\varepsilon)$ contains an element of $\mathbf{Q}$ other than x. We see that the derived set can be *much* larger than the set itself (here the derived set of a countable set is uncountable). The Density Theorem also tells us that $(\mathbf{R}\backslash\mathbf{Q})' = \mathbf{R}$. Note that two sets can have the same derived set without being the same. There are no "antiderivatives" here.

Our goal is not to study derived sets for their own sake, but we will prove one theorem before we move on. We write S'' for $(S')'$.

THEOREM 7.4: *For any set S, $S'' \subseteq S'$.*

PROOF: We will examine this proof by the forward-backward method. We will use the alternative form of the definition given in Statement (6). This is a set containment problem, and so we know that it has to begin and end like this:

▽▽

Let $x \in S''$.

⋆ ⋆ ⋆

Then $x \in S'$.

△△

The definition of the conclusion ($x \in S'$) includes a universally quantified variable, ε. We must introduce it at the beginning, as we insert the rest of the definition at the end:

▽▽

Let $x \in S''$.

$\hookrightarrow$ Let $\varepsilon > 0$ be given.

⋆ ⋆ ⋆

$\hookrightarrow$ $(x - \varepsilon, x + \varepsilon) \cap S \backslash \{x\} \neq \emptyset$.

Then $x \in S'$.

△△

The phrase "$x \in S''$" has a definition ...

▽▽

Let $x \in S''$.

Let $\varepsilon > 0$ be given.

$\hookrightarrow$ Since $x \in S''$, x is a cluster point of S'.

⋆ ⋆ ⋆

$(x - \varepsilon, x + \varepsilon) \cap S \backslash \{x\} \neq \emptyset$.

Then $x \in S'$.

△△

...as does "x is a cluster point of S'"...

Let $x \in S''$.

Let $\varepsilon > 0$ be given.

Since $x \in S''$, x is a cluster point of S'.

$\hookrightarrow (x-\varepsilon, x+\varepsilon) \cap S' \backslash \{x\} \neq \emptyset$.

⋆ ⋆ ⋆

$(x-\varepsilon, x+\varepsilon) \cap S \backslash \{x\} \neq \emptyset$.

Then $x \in S'$.

We seem to be almost done, but the set we *know* to be nonempty is not the one we *wish* to be nonempty. Since we have found that something exists [an element of $(x-\varepsilon, x+\varepsilon) \cap S' \backslash \{x\}$], this is a good time to draw a sketch. Let y be the element of $(x-\varepsilon, x+\varepsilon) \cap S' \backslash \{x\}$ that we are guaranteed. Note in particular that $y \in S'$.

(In this picture, we have put y to the right of x. We should be careful that nothing in our proof makes use of this since it might not always be the case.) The ε-interval around x is a neighborhood of y, and so contains an ε-neighborhood of y. Since y is a cluster point of S, we know something about ε-neighborhoods of y. We select a new value of ε by examining the drawing, and, with a brief observation, find that we're done:

Let $x \in S''$.

Let $\varepsilon > 0$ be given

Since $x \in S''$, x is a cluster point of S'. not nec. need to prove

$(x-\varepsilon, x+\varepsilon) \cap S' \backslash \{x\} \neq \emptyset$.

$\hookrightarrow$ Let $y \in (x-\varepsilon, x+\varepsilon) \cap S' \backslash \{x\}$.

• $\hookrightarrow$ Let $\varepsilon_1 = \min\{y-(x-\varepsilon), (x+\varepsilon)-y\}$.•

$\hookrightarrow$ Since $y \in S'$, $(y-\varepsilon_1, y+\varepsilon_1) \cap S \backslash \{y\} \neq \emptyset$.

$\hookrightarrow$ Let $z \in (y-\varepsilon_1, y+\varepsilon_1) \cap S \backslash \{y\}$.

$\hookrightarrow$ Then $z \in (x-\varepsilon, x+\varepsilon) \cap S \backslash \{x\}$.

$(x-\varepsilon, x+\varepsilon) \cap S \backslash \{x\} \neq \emptyset$.

Then $x \in S'$. ∎

Most of the work in this proof was suggested by the picture. The moment in the proof where we find what we are looking for (which happens in the third from last line) pops in quite easily. We do have to remember where we want the proof to go to see that the statement is useful, but it is not difficult to make this connection.

The line marked by •• is where we *might* have erroneously used the position of y in the picture. The ε_1-interval around y should be contained in the ε-interval around x. To accomplish this, ε_1 should be the distance from y to the end of the ε-interval. In the picture, this is $x + \varepsilon - y$ since y is nearer the right end of the interval. If y were to the left of x, this would not be true, and $x + \varepsilon - y$ would be too big. By choosing ε_1 as we have, we allow for both possibilities.

Since it is not always true that $X' \subseteq X$, not every set is a derived set. Theorem 7.4 suggests that derived sets might be more well-behaved than are typical sets. We will see just which sets can be derived sets in Exercise 7.5.9.

EXERCISES 7.5

1. There is an error in the proof of Theorem 7.4. Find it and fix it.

2. Show that if $S \subseteq T$, then $S' \subseteq T'$.

3. Find the derived sets of the following sets:

   (a) $\{2 + \frac{(-1)^n}{n} : n \in \mathbf{N}\}$

   (b) $\{\sin(n) : n \in \mathbf{N}\}$

   (c) $\{\frac{1}{n} + \frac{1}{m} : n, m \in \mathbf{N}\}$

4. (a) Construct a set for which $S'' \neq S'$.

   (b) Construct a set for which $S''' \neq S''$.

5. Show that adding finitely many points to a set or deleting finitely many points from a set does not change its derived set.

6. (a) Show that $(A \cup B)' = A' \cup B'$ (Hint: Use the forward-backward method!)

   (b) Show that the relationship in (a) holds for a union of finitely many sets.

   (c) Show that $\bigcup_{\alpha \in \mathcal{A}} (A_\alpha)' \subseteq \left(\bigcup_{\alpha \in \mathcal{A}} A_\alpha\right)'$ in any event, but that this containment might be strict if $\mathcal{A}$ is infinite.

   (d) Show that $\left(\bigcap_{\alpha \in \mathcal{A}} A_\alpha\right)' \subseteq \bigcap_{\alpha \in \mathcal{A}} (A_\alpha)'$ and that this inclusion might be strict, even if $\mathcal{A}$ is finite.

7. Why does, the following *not* establish that $\mathbf{Q}' = \mathbf{R}$? *By the Density theorem, if x is any real number and $\varepsilon > 0$*, then $(x-\varepsilon, x+\varepsilon) \cap \mathbf{Q} \neq \emptyset$.

8. Show that S is dense in $\mathbf{R}$ if and only if $S' = \mathbf{R}$.

9. (a) Construct a set whose derived set is $\{1, 2, 3\}$.

   (b) Is there a set whose derived set is $\mathbf{Q}$? (Consider Theorem 7.4.)

   (c) If $x \in S$ and x is not a cluster point of S, then x is called an **isolated point** of S. Show that x is an isolated point of S if and only if there is an $\varepsilon > 0$ so that $(x - \varepsilon, x + \varepsilon) \cap S = \{x\}$.

   (d) Suppose that $\{x_\alpha\}$ is the collection of isolated points of a set S. Show that there exist positive numbers $\{\varepsilon_\alpha\}$ so that the ε_α-intervals around x_α are mutually disjoint.

   (e) Show that a set can have at most countably many isolated points.

   (f) Suppose S is a set that contains all of its cluster points. Show that S is the derived set of some (possibly different) set.

7.6 THE BOLZANO-WEIERSTRASS THEOREM

Are there circumstances under which we can guarantee that a set has a cluster point? We have found that a set with a cluster point must be infinite, but that there are infinite sets with no cluster points ($\mathbf{N}$, for example). The natural numbers and the set H in Example 7.5.2 are very similar in some ways. If we look at very small pieces of the number line, H and $\mathbf{N}$ seem much the same. Looking very closely at any element of either set, we see no *other* point of the set. But H has a cluster point, and $\mathbf{N}$ does not. The difference must lie in the *larger* structure of the sets; it must be some property of the whole set. For instance, we might note that H is bounded, while $\mathbf{N}$ is not. It happens that this is just what we need.

THEOREM 7.5: (The Bolzano-Weierstrass Theorem) *If $\mathbf{F}$ is an Archimedean ordered field in which the Nested Intervals property holds, then any bounded, infinite subset of $\mathbf{F}$ has a cluster point.*[2]

The proof that follows is as much a piece of history as a piece of mathematics. This can be troublesome if the years have polished an argument to the point where the motivation for it can't be seen anymore. We are

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[2] It is best to remember the Bolzano-Weierstrass theorem as: *Every bounded, infinite set has a cluster point*, but of course we must state our result as we have to keep it in the context of the Big Theorem.

trying to show that a cluster point exists, and certainly the best way to do that is to *find one*. We have essentially only one tool available for this task—the Nested Intervals property. What we must do, then, is construct a nest of closed, bounded intervals whose intersection consists of a point that is a cluster point of the set in question. This point will be in every interval in the nest (since it is in the intersection) and if an interval contains a cluster point of a set, it very likely contains infinitely many points of that set (draw a sketch to convince yourself of this). We should be looking for a nest of intervals, each of which contains infinitely many points of the set. Keeping this in mind makes the proof much easier to follow.

PROOF: Let S be a bounded, infinite set, and let b_0 be an upper bound for S and a_0 a lower bound, so that $S \subseteq [a_0, b_0] = I_0$. Let I_L be the left half of I_0, and I_R the right half (to be precise, $I_L = [a_0, (a_0 + b_0)/2]$ and $I_R = [(a_0 + b_0)/2, b_0]$). Now one (or both) of $S \cap I_L$ or $S \cap I_R$ is infinite, since otherwise $S = (S \cap I_L) \cup (S \cap I_R)$ would be finite. Let $I_1 = I_L$ if $S \cap I_L$ is infinite and $I_1 = I_R$ if $S \cap I_L$ is finite, and let $S_1 = S \cap I_1$. Now S_1 is an infinite set contained in the interval I_1. The process by which we obtained I_1 can be repeated to find another interval, I_2, which is either the left or right half of I_1, and is such that $S_2 = S \cap I_2$ is infinite. Continuing in this way, we find a nest of intervals: $I_1 \supseteq I_2 \supseteq \ldots$, each the left or right half of the previous one and such that $S_n = S \cap I_n$ is infinite for all n.

By the Nested Intervals property, $\bigcap_n I_n \neq \emptyset$, and since the infimum of the lengths of the intervals I_n is 0, there is a real number x so that $\bigcap_n I_n = \{x\}$. We will show that x is a cluster point of S. Let $\varepsilon > 0$ be given. There is an interval I_{n_0} whose length is less than ε (all but finitely many of them have this property). Since the length of I_{n_0} is less than ε and $x \in I_{n_0}$, we have $I_{n_0} \subseteq (x - \varepsilon, x + \varepsilon)$ (by Theorem 4.20). Now $S_{n_0} \subseteq I_{n_0} \subseteq (x - \varepsilon, x + \varepsilon)$, and S_{n_0} is infinite (this is the way the intervals I_n were chosen). It follows that $S \cap (x - \varepsilon, x + \varepsilon)$ is infinite and that x is a cluster point of S. ∎

EXERCISES 7.6

1. Where is the Archimedean property used in the proof of the Bolzano-Weierstrass theorem?

2. Consider the comments preceding the proof of the Bolzano-Weierstrass theorem. Under what circumstances could an interval contain a cluster point of a set *without* containing infinitely many points of the set?

3. If a set S has no cluster points, show that S must be either finite or unbounded.

4. (a) Give an example of an infinite set with no cluster point.

   (b) Give an example of an infinite set having the property that its intersection with any set of the form $[-n, n]$ is finite.

   (c) Suppose S is a set with the property that $S \cap [-n, n]$ is finite for each $n \in \mathbf{N}$. Show that S is countable.

   (d) Show that every uncountable subset of the real line has a cluster point.

5. (a) Show that the Bolzano-Weierstrass theorem fails in the field of formal rational functions.

   (b) Show that the Bolzano-Weierstrass theorem fails in *any* ordered field that is not Archimedean.

6. (a) Let S be a bounded, infinite set. Show that S' is also bounded.

   (b) If S is as in (a), we define the **limit superior** of S by $\limsup S = \sup S'$. Show that $a = \limsup S$ if and only if $(a - \varepsilon, \infty) \cap S$ is infinite for all $\varepsilon > 0$ and $(a + \varepsilon, \infty) \cap S$ is finite for all $\varepsilon > 0$.

   (c) If S is bounded above, show that $\limsup S$ is a cluster point of S (first convince yourself that this is not just part of the definition).

   (d) If S is bounded below, the **limit inferior** of S is given by $\liminf S = \inf S'$. State and prove results similar to (b) and (c).

   (e) Show that it is possible for S' to be bounded even if S is not.

   (f) If S is such that S' is not bounded above, we say $\limsup S = \infty$. Show that if $\limsup S = \infty$, then $S \cap (a, \infty)$ is infinite for all a, but that this condition is not sufficient to guarantee that $\limsup S = \infty$.

   (g) If $S' = \emptyset$, what are $\limsup S$ and $\liminf S$?

   (h) Is there any relationship between these uses of the words "lim inf" and "lim sup" and the usage in Exercise 1.15.9? (For each $x \in S$, consider the set $S_x = \{y : y < x\}$. Is there a relationship between $\limsup S$ and $\limsup\{S_x : x \in S\}$?)

7. (a) If S is any bounded infinite set, show that $\liminf S \leq \limsup S$.

   (b) If in addition S has more than one cluster point, show that $\liminf S < \limsup S$.

8. The following "proof" contains at least two serious errors. Find them and explain why they are errors. Give an example to show that the "theorem" is false.

"THEOREM" Every nonempty set that is bounded above contains a point that is a cluster point of the set.

"PROOF" Let A be a nonempty set that is bounded above. Since it is bounded above, it has a supremum, say u. By Theorem 5.3, for every $\varepsilon > 0$, there is an element of A, say a_ε, with $a_\varepsilon > u - \varepsilon$. Then $a_\varepsilon \in (u - \varepsilon, u + \varepsilon)$, and so u is a cluster point of A. Since u is the supremum of A, $u \in A$, and so A contains one of its cluster points.

9. (a) Show that a bounded set having exactly one cluster point is denumerable. (Hint: If S is such a set and c is the cluster point, consider the sets $S \cap \left((-\infty, c - \frac{1}{n}) \cup (c + \frac{1}{n}, \infty)\right)$. How many elements can these sets have?)

   (b) Show that the assumption in part (a) that S is bounded is unnecessary.

   (c) Show that a set having finitely many cluster points is denumerable.

   (d) Is a set having denumerably many cluster points necessarily denumerable?

   (e) Is a set having uncountably many cluster points necessarily uncountable?

7.7 CLOSING THE LOOP

Now we wish to show that the Bolzano-Weierstrass theorem and the Archimedean property together imply the Least Upper Bound property.[3] This proof is a construction and as such may not be as transparent as some other proofs. Before we begin, we will get an idea what it is we want to accomplish, but with constructions, one must often adopt the attitude "Follow along, and it will work out in the end."

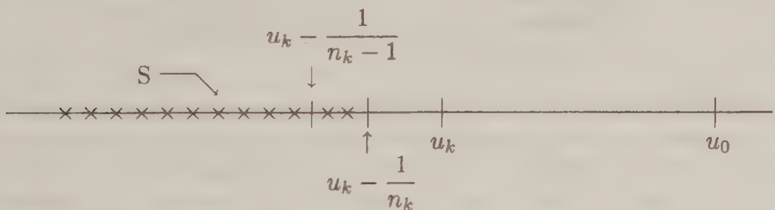
THEOREM 7.6: *If* **F** *is an Archimedean ordered field in which the Bolzano-Weierstrass theorem holds, then the Least Upper Bound property also holds in* **F**.

You will pick out where the Archimedean Property is used in this proof in Exercise 7.7.1. We must begin with a nonempty set that is bounded above, and we wish to construct an auxiliary set of some sort having a

[3] A note on the structure of the Big Theorem: The Archimedean property [which is stated in part (b) of the Big Theorem] implies itself [stated in part (c)].

cluster point that is the supremum of the original set. Since a set can have only one supremum, we should construct a set with only one cluster point. Recall that the set $H = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ has only one cluster point, and so we might try to make our auxiliary set should look something like H. This may not be much to go on, but keeping it in mind will make the steps of the proof a bit more reasonable.

PROOF: Let S be a nonempty set that is bounded above, and let u_0 be an upper bound for S. There is a natural number n so that $u_0 - n$ is an upper bound for S but $(u_0 - n) - 1$ is not. Let $u_1 = u_0 - n$. (Note that u_1 is an upper bound for S but $u_1 - 1$ is not— we strongly suspect that $u_1 - 1 < \sup S \leq u_1$.) If $u_1 - (1/n)$ is not an upper bound for S for any natural number n, then $u_1 = \sup S$ (Exercise 6.1.10). If this is the case, we are done. Otherwise, $\{n : u_1 - (1/n) \text{ is an upper bound for } S\} \neq \emptyset$, and, by well-ordering, has a least element. Call this number n_1. Note that $n_1 \neq 1$ and that $u_1 - (1/n_1)$ is an upper bound for S but $u_1 - (1/(n_1 - 1))$ is not (we have further narrowed down where $\sup S$ might be). Let $u_2 = u_1 - (1/n_1)$ and begin again: If $u_2 - (1/n)$ is not an upper bound for S for any n, then $u_2 = \sup S$, and we are done. Otherwise, let n_2 be the least natural number so that $u_2 - (1/n_2)$ is an upper bound for S and let $u_3 = u_2 - (1/n_2)$. These steps are now repeated. Here is a picture of one step (remember, though, that S may not be as simple as it appears in this diagram).



This process might end after finitely many steps [after saying "$u_k - (1/n)$ is not an upper bound for S for any value of n, and so $u_k = \sup S$ and we are done"]. On the other hand, it might never end. We may assume that it doesn't. We have made sets $U = \{u_k : k \in \mathbf{N}\}$ and $N = \{n_k : k \in \mathbf{N}\}$, so that

(1) u_k is an upper bound for S for every k;

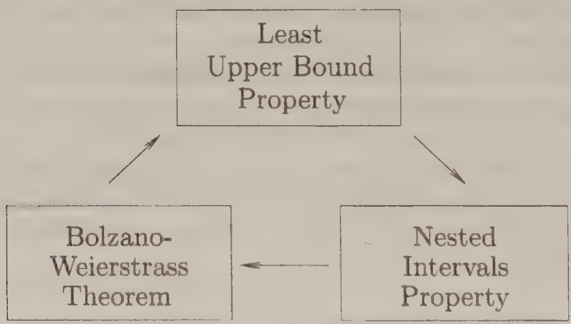
(2) $u_{k+1} = u_k - (1/n_k)$;

(3) $u_k - (1/(n_{k-1}))$ is not an upper bound for S for any k.

Now $n_k > n_{k-1}$ for all k, and so $\sup N = \infty$ and $\inf\{(1/n_k) : k \in \mathbf{N}\} = 0$. Since $u_{k+1} < u_k$ for all k, u_1 is an upper bound for U. Each element of S is a lower bound for U. Thus U is a bounded, infinite set, and so it has a

cluster point by the Bolzano-Weierstrass theorem. Call the cluster point u. Where can u be? Suppose x is such that $x < s$ for some $s \in S$ and let $\varepsilon = s - x$. Then there can be no elements of U in $(x - \varepsilon, x + \varepsilon)$ since all elements of U are greater than s. Therefore, if x is not an upper bound for S, x is not a cluster point of U. It follows that u is an upper bound for S. By Lemma 10.2 (!?!—the terminology of Chapter 10 will make this lemma easier to prove), $u < u_k$ for all k. Let $d < u$ and let $k \in \mathbf{N}$ be such that $1/(n_k - 1) < u - d$. Remember that $u_k - (1/(n_k - 1))$ is not an upper bound for S. Since $d < u - (1/(n_k - 1)) < u_k - (1/(n_k - 1))$, we see that d is not an upper bound for S. Since u is an upper bound for S and d is *not* an upper bound for S whenever $d < u$, we have $u = \sup S$. ∎

So far, we have completed this much of the Big Theorem:



Let us reflect for a moment on the Big Question—the difference between the rational and real numbers. We showed at the end of Chapter 6 that the rational numbers do not have the Nested Intervals property (by finding a carefully chosen nest). Observe that the set of left endpoints of the intervals in that nest (thought of as a subset of $\mathbf{Q}$) constitute both a nonempty set that is bounded above but has no supremum and a bounded, infinite set with no cluster point (you will verify these claims in Exercise 7.7.3). By reinterpreting the same example, we see that neither the Least Upper Bound property nor the Bolzano-Weierstrass theorem holds in the rational numbers.

EXERCISES 7.7

1. (a) Verify the statement in the proof of Theorem 7.6: "$n_k > n_{k-1}$ for all k, and so $\sup N = \infty$, and $\inf\{1/n_k : k \in \mathbf{N}\} = 0$."

   (b) Find where the Archimedean property is used in the proof of

Theorem 7.6.

(c) Show that the process described in this proof ends after finitely many steps if and only if the difference between u_0 and u is rational.

2. Suppose U and S are sets with the properties: (i) each element of U is an upper bound for S; (ii) for any $\varepsilon > 0$, there are elements u of U and s of S with $|u - s| < \varepsilon$.

   (a) Show that $\inf U = \sup S$.

   (b) Find where this can be inserted as a lemma in the proof of Theorem 7.6.

3. Verify the claims made in the final paragraph of the chapter.

Chapter 8

Topology of the Real Numbers

8.1 OPEN SETS

The importance of intervals in our investigations suggests it might be worthwhile to generalize their properties. We will list some things we know about open and closed intervals, and then create definitions of "open" and "closed" that can apply to sets other than intervals. This is a roundabout way of doing things, but it will be very fruitful. Just about everything we have done has had something to teach us about open intervals. For instance:

(1) An open interval is a set of the form $\{x : a < x < b\}$ for some numbers a and b (a may be replaced by $-\infty$ or b by ∞).

(2) An open interval doesn't contain its supremum or infimum, even if it has one.

(3) An open interval is a neighborhood of each of its points.

(4) An open interval (one that isn't the whole line) has a cluster point that it doesn't contain.

(5) An open interval is uncountable.

We want to make of one of these statements our definition of "open set." Which would work best? The first is much too specific. If we took it as our definition, we wouldn't get any open sets other than the intervals themselves. The second is better, but there are sets that would be open if we adopted this as our definition that we might not want. The set $(0, 1] \cup [2, 3)$ doesn't contain its supremum or infimum, but it doesn't look much like an open interval when we look near the points 1 and 2, and we probably shouldn't call it open. The fifth property is *too* general. The interval $[1, 2]$ is uncountable, but we probably don't want it to be open. This leaves us with the third statement, which we take as our definition:

DEFINITION 8.1: A set is **open** if it is a neighborhood of each of its points.

If we combine this definition with the definition of neighborhood, we find that a set A is open if, for every $x \in A$, there is an $\varepsilon_x > 0$ with $(x - \varepsilon_x, x + \varepsilon_x) \subseteq A$. Take careful note of the quantifiers.

EXAMPLES 8.1: 1. Open intervals are open sets. The whole real line is open.

2. The empty set is open. Since $\emptyset$ contains no points at all, it does not contain any point of which it is *not* a neighborhood.

3. Let $A = (0,1) \cup (3,4)$. This is a good time to recall a couple of our basic techniques of proof. The definition of open is universally quantified, and so to prove that the set A is open, we should begin by giving a name to an element of A: Let $x \in A$. Now we need to show that the definition of open is satisfied for *this particular* x, that is, we must show that A is a neighborhood of x. Now A is a given as the union of two sets, which means that the *hypothesis* of this statement $(x \in A)$ can be rewritten: $x \in (0,1)$ or $x \in (3,4)$. The "or" in the hypothesis indicates a proof by cases:

Case 1: $x \in (0,1)$. We know that $(0,1)$ is a neighborhood of x since $(0,1)$ is an open interval. Exercise 4.10.1 says that if U is a neighborhood of x and $U \subseteq V$, then V is a neighborhood of x. But $(0,1) \subseteq A$, and so A is a neighborhood of x. Notice that **Case 2**—$x \in (3,4)$—is resolved in the same way since $(3,4)$ is an open interval and $(3,4) \subseteq A$.

4. $B = [0,1)$ is *not* open. Every ε-neighborhood of $0 \in B$ contains a point outside B (to be specific, the point $-\varepsilon/2$).

We didn't use all we know about open intervals in Example 3 (for instance, we didn't refer to endpoints). Because of this, we can deduce much more from the proof than we originally stated. The only property of open intervals used was that they are neighborhoods of each of their points. But this is just the definition of an open set! By *precisely* the same proof, we obtain:

THEOREM 8.2: *The union of two open sets is open.* ∎

The crucial step in Example 3 was the application of Exercise 4.10.1, which says that if $A \subseteq B$ and A is a neighborhood of x, then B is a neighborhood of x. If B is formed by the union of A with *any* collection of sets, then $A \subseteq B$. We may use the same proof again to establish that:

THEOREM 8.3: *The union of any collection of open sets is open.* ∎

It is *not* true that the *intersection* of any collection of open sets is open: Let $U_n = (-1/n, 1/n), n = 1, 2 \ldots$. Then U_n is open for each n, but $\bigcap_n U_n = \{0\}$, which is not open. On the other hand, we have shown (in Exercise 4.10.2) that the intersection of a *finite* collection of neighborhoods of a point x is a neighborhood of x. It follows that

THEOREM 8.4: *The intersection of any finite collection of open sets is open.* ∎

EXERCISES 8.1

1. To use Exercise 4.10.1 in the last step of the proof of Theorem 8.2, we need to know that $A \subseteq A \cup B$. But this holds no matter what B is. Does the proof show, then, that *any* set that contains an open set is open?

2. Prove Theorem 8.4.

3. Do an induction to show that the union of a finite collection of open sets is open.

4. Show in detail that $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1 + \frac{1}{n}\right) = [0, 1]$.

5. The definition of open set is quantified: "$\forall x \exists \varepsilon \ldots$." Suppose this is reversed to "$\exists \varepsilon \forall x \ldots$." Show that the only nonempty set satisfying this new condition is the whole real line.

8.2 GENERAL TOPOLOGIES

We have found that open sets have certain properties, which we bring together to form an important definition:

DEFINITION 8.5: Let X be a set and $\mathcal{T}$ a collection of subsets of X such that:

(i) $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$.
(ii) The union of any collection of sets from $\mathcal{T}$ is in $\mathcal{T}$.
and (iii) The intersection of any finite collection of sets from $\mathcal{T}$ is in $\mathcal{T}$.

Then $\mathcal{T}$ is called a **topology on** X (this explains the title of the chapter). The pair $(X, \mathcal{T})$ is called a **topological space**, and the elements of $\mathcal{T}$ (which are themselves sets, remember) are called **open subsets of** X.

Our goal here is not to study topology for its own sake (there are exercises

throughout the chapter about it), but if you look at an introductory topology text after finishing this one, you will find many familiar words and ideas. There is one idea from topology, more an attitude than a theorem, that will be of some importance to us. A proof or definition that is *topological*, that is, one that uses only facts about open sets, is usually preferable to one that uses other structures (such as order, algebra, or distance). There is no specific reason for this, though any proof we can do "topologically" will not have to be redone when we go on to more abstract topics (in the same way that our early proof that there is only one additive identity in a field need never be done again). There is also a certain "elegance" in doing more with less. This lofty notion will become more palatable when we discover that many proofs are easier when done with more abstract tools!

EXERCISES 8.2

1. (a) Let $S = \{1, 2, 3\}$ and let $\mathcal{T} = \{\emptyset, \{1, 2, 3\}, \{1, 2\}, \{3\}\}$. Show that $\mathcal{T}$ is a topology on S.

   (b) Let $S = \{1, 2, 3\}$, and let $\mathcal{T} = \{\emptyset, \{1, 2, 3\}, \{1, 2\}, \{1, 3\}\}$. Show that $\mathcal{T}$ is *not* a topology on S.

   (c) Let $S = \{1, 2\}$. Find all *sets of subsets* of S (there are 16 of them). Which are topologies?

   (d) Repeat (c) with $S = \{1, 2, 3\}$.

   (e) Show that $\mathcal{T} = \{S \subseteq \mathbf{R} : \mathbf{R} \backslash S \text{ is finite}\} \cup \{\emptyset\}$ is a topology on $\mathbf{R}$. This is called the **finite complement** topology.

   (f) Notice in (e) that the complement of $\emptyset$ is *not* finite. Why is it included in $\mathcal{T}$?

   (g) If X is any set, show that $\mathcal{T} = \{S \subseteq X : X \backslash S \text{ is finite}\} \cup \{\emptyset\}$ is a topology on X.

2. (a) Show that a set is dense in the real line if its intersection with any open set is nonempty (the definition of "dense" requires only that the intersection of the set with any open *interval* is nonempty).

   (b) Show that the natural numbers are not dense in the real line.

   (c) Let $\mathcal{T}$ consist of $\emptyset$, $\mathbf{R}$, and all sets of the form (a, ∞) for some $a \in \mathbf{R}$. Show that $\mathcal{T}$ is a topology on $\mathbf{R}$.

   (d) Show that the natural numbers *are* dense in the real line (using the definition given in (a) of this problem) if it is given this topology.

   (e) Let $\mathcal{T}$ consist of $\emptyset$, $\mathbf{R}$, and all sets of the form $[a, \infty)$ for some $a \in \mathbf{R}$. Show that $\mathcal{T}$ is *not* a topology on $\mathbf{R}$.

3. (a) If X is a set, show that $P(X)$ and $\{\emptyset, X\}$ are topologies on X. The former is called the **discrete topology**. The latter, in contrast, is called the **indiscrete topology**. Notice that the discrete topology, which consists of *every* subset of X, is the "biggest" topology on X (in the sense that it contains the most sets), while the indiscrete topology is the smallest collection of subsets that can possibly be a topology, since it contains only those two sets that any topology *must* contain.

   (b) Give an example of a topological space having a finite, dense subset (you must give a set and a topology).

   (c) Show that *any* nonempty set is dense in *any* space having the indiscrete topology.

   (d) Describe the dense subsets in a space having the discrete topology.

   (e) If X is finite, show that the finite complement topology on X is the same as the discrete topology. (Think carefully about what "the same" means in this setting. On the simplest level, a topology is a *set*.)

   (f) If X is infinite, show that the finite complement topology on X is *not* the same as the discrete topology.

8.3 CLOSED SETS

Here are some of the things we have learned about *closed* intervals:

(1) A closed interval is a set of the form $\{x : a \leq x \leq b\}$, $\{x : a \leq x < \infty\}$, or $\{x : -\infty < x \leq b\}$ for some a and b, or is the whole line.

(2) A closed interval contains its supremum and/or infimum, if it has one.

(3) A closed interval contains all its cluster points.

(4) The Nested Intervals property holds for closed, bounded intervals.

If we are looking for a general definition of "closed set," we may reject the first and second of these as before: (1) is too specific, and (2) would admit "closed" sets we probably don't want. The intervals in the Nested Interval property had to be closed *and* bounded, but there are unbounded closed intervals. Condition (4) is too restrictive (although closed, bounded sets will be very important to us later). This leaves us with:

DEFINITION 8.6: A set is **closed** if it contains all its cluster points (that is, if $S' \subseteq S$).

EXAMPLES 8.3: 1. Closed intervals and the whole real line are closed sets.

2. The empty set is closed since it has no cluster points.

3. $A = [0,1] \cup [3,4]$ is closed.

4. $B = [0,1)$ is *not* closed since it has $x = 1$ as a cluster point, but $1 \notin B$. This set is also not open. **"Not closed" is not the same as "open."**

5. $H = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ is *not* closed since $0 \notin H$ is a cluster point of H.

6. $S = H \cup \{0\}$ *is* closed.

These examples suggest some important results. The observation in the second example may be made into a theorem, telling us, among other things, that finite sets are closed:

THEOREM 8.7: *A set having no cluster points is closed.* ∎

Theorem 8.3 does not hold for closed sets: $U_n = [1/n, 2]$ is closed for each $n \in \mathbf{N}$, but $\bigcup_n U_n = (0, 2]$, which is not closed. But all is not lost. By Exercise 7.5.6.a, $(A \cup B)' = A' \cup B'$. If $A' \subseteq A$ and $B' \subseteq B$, then $(A \cup B)' = A' \cup B' \subseteq A \cup B$, and so $A \cup B$ is closed. By induction, we have:

THEOREM 8.8: *The union of finitely many closed sets is closed.* ∎

Exercise 7.5.6.a does not hold for infinite unions, but Exercise 7.5.6.c says that $(\bigcap_\alpha U_\alpha)' \subseteq \bigcap_\alpha (U_\alpha)'$, which leads us to the following:

THEOREM 8.9: *The intersection of any collection of closed sets is closed.* ∎

So far we have seen that:

(i) The empty set and the whole real line are closed.
(ii) The intersection of any collection of closed sets is closed.
and (iii) The union of a finite collection of closed sets is closed.

This would seem to be a nice mirror image of the similar results for open sets. We could define a topology beginning with closed sets, but it is not often done that way. This symmetry certainly suggests some connection between open sets and closed sets, though. The connection is actually

very strong, as seen in the following theorem. Since this is an "if-and-only-if" theorem, we could have defined either "open" or "closed" and used this theorem as the definition of the other. *Theorem 8.10 is the traditional definition of "closed."*

THEOREM 8.10: *A set is open if and only if its complement is closed.*

PROOF: We will use the forward-backward method on the "only if" part of this proof.

▽▽▽

Let A be open.

⋆ ⋆ ⋆

Then $C(A)$ is closed.

△△△

We insert the definitions of "open" and "closed":

▽▽▽

Let A be open.

$\hookrightarrow$ A is a neighborhood of each of its points.

⋆ ⋆ ⋆

$\hookrightarrow$ $C(A)$ contains all its cluster points.

Then $C(A)$ is closed.

△△△

The definition of "closed" is a universally quantified statement about cluster points. We need to give a name to a specific cluster point of $C(A)$ and prove something about that particular point:

▽▽▽

Let A be open.

A is a neighborhood of each of its points.

$\hookrightarrow$ Let x be a cluster point of $C(A)$.

⋆ ⋆ ⋆

$\hookrightarrow$ $x \in C(A)$.

$C(A)$ contains all its cluster points.

Then $C(A)$ is closed.

△△△

We insert the definitions of "cluster point" and "complement":

▽▽

Let A be open.

A is a neighborhood of each of its points.

Let x be a cluster point of $C(A)$.

$\hookrightarrow$ Every neighborhood of x contains infinitely many points of $C(A)$.

⋆ ⋆ ⋆

$\hookrightarrow x \notin A$.

$x \in C(A)$.

$C(A)$ contains all its cluster points.

Then $C(A)$ is closed.

△△

We make an observation combining two of our statements, remember the definition of open sets, and find that we are done:

▽▽

Let A be open.

A is a neighborhood of each of its points.

Let x be a cluster point of $C(A)$.

Every neighborhood of x contains infinitely many points of $C(A)$.

$\hookrightarrow$ If A were a neighborhood of x, it too would contain infinitely many points of $C(A)$.

$\hookrightarrow$ This is not the case, and so A is not a neighborhood of x.

$\hookrightarrow$ But then, since A is open ...

$x \notin A$.

$x \in C(A)$.

$C(A)$ contains all its cluster points.

Then $C(A)$ is closed.

△△

We now continue with the other half of the proof. Suppose $C(A)$ contains all of its cluster points [that is, $C(A)$ is closed], and let $x \in A$. We must show that A is a neighborhood of x (to show that A is open). Since $x \notin C(A)$, x is not a cluster point of $C(A)$. Therefore there is a neighborhood of x that has no point in common with $C(A)$. This neighborhood is contained in A, and since A *contains* a neighborhood of x, A *is* a neighborhood of x. ∎

In everyday language, things normally can't be both open and closed. Furthermore, if an object *can* be either open or closed, it usually must

be one or the other ("ajar" notwithstanding). In mathematics, **"closed" does NOT mean "not open."** The empty set and the real line are each *both* open and closed (we will see later that they are the only sets with this property), while the set $[0, 1)$ is *neither* open nor closed. We should not read into the words "open" and "closed" meanings other than those we have given them.

EXERCISES 8.3

1. (a) Show that a nonempty closed, bounded set contains its supremum and infimum.

   (b) Is the converse of this true?

2. The **closure** of a set S, denoted S^-, is the intersection of all closed sets *containing* S. The **interior** of S, denoted S°, is the union of all open sets *contained in* S. The **boundary** of S is $S^- \setminus S^\circ$, and is denoted ∂S.

   (a) Show that S^- is the smallest closed set containing S, and S° is the largest open set contained in S. (The meanings of "smallest" and "largest" were discussed in Exercise 1.15.7.)

   (b) Show that $x \in \partial S$ if and only if every neighborhood of x contains points of S and of $C(S)$ (elements of ∂S are called **boundary points**).

   (c) Give an example of a countable set with an uncountable boundary.

   (d) Let $I = (0, 1)$. Find I^-, I°, and ∂I. Prove your results.

   (e) Repeat (d) for: $I = [0, 1)$, $J = [0, 1]$, and $H = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$.

   (f) Show that S is closed if and only if $S = S^-$, and S is open if and only if $S = S^\circ$.

   (g) Show that ∂S is closed.

   (h) Explain why the following "proof" of the first half of part (a) is *invalid* [you should pay especially close attention if this was your answer to part (a)!]:

   > Let C be the smallest closed set containing S. The intersection of all closed sets containing S can't be any larger than C because C is one of the sets being intersected. Thus $S^- \subseteq C$. On the other hand, S^- is a closed set containing C, and C is the smallest such set, and so $C \subseteq S^-$, and thus $C = S^-$.

   (i) The flaw in part (h) is very similar at its root to the flaw in the following "proof" that 1 *is the largest natural number*(!) Explain.

   > Let n be a natural number greater than 1. Then $n^2 > n$, and so n is *not* the largest natural number. Consequently 1 is the largest natural number.

(Notice that the argument here is sound, and so the flaw must have something to do with what happened *before* the argument began.)

3. (a) Find the interior, closure, and boundary of the rational numbers if the real numbers are given the standard topology.

   (b) Find the interior, closure, and boundary of the rational numbers if the real numbers are given the topology in the Exercise 8.2.2.c.

4. (a) Let S be a bounded set. Show that $\inf S$ and $\sup S$ are boundary points of S.

   (b) If S is nonempty and S has no boundary points, show that S is unbounded.

   (c) Show that an interior point of a set can't be a boundary point.

5. (a) Construct a set for which the set, its closure, the complement of its closure, and the closure of the complement of its closure are all different.

   (b) Suppose we repeat this over and over: closure $\rightarrow$ complement $\rightarrow$ closure $\rightarrow \ldots$. Is there a limit to how many *different* sets can be obtained in this way? (Hint: If A is the *closure* of an *open* set, show that $C(C(A)^-)^- = A$. This is called **Kuratowski's problem**. The answer is 14. Try to construct a set for which this number is attained.)

8.4 THE STRUCTURE OF OPEN SETS

Looking at the previous theorem, we might guess that the study of closed sets and the study of open sets amount to much the same thing. In view of the examples, though, we see there might be some differences. A nonempty open set, since it contains an open interval, is uncountable. Closed sets can have all sorts of cardinalities. Are closed sets somehow more complicated than open sets? We will find that, in a way, they are. We show now that open sets are quite predictable in their structure, and we will find later (see Exercise 8.4.7) that closed sets can be very peculiar. This is a long proof, but it uses a variety of techniques and is a good detective story.

THEOREM 8.11: *A nonempty set S is open if and only if there is a countable collection of mutually disjoint open intervals $\{U_1, U_2, \ldots\}$ such that $S = \bigcup_n U_n$.*

PROOF: The "if" part is already established since *any* union of open sets is open. There is much to do in the other direction. We must construct a

collection of intervals from the given set S. Our first step is to associate with each point in S the largest open interval containing it and contained in S. How can we go about this? Suppose S is a single open interval, say $(0,1)$, and consider a point in it, say $1/3$. Can we distinguish points less than $1/3$ that are *in* the interval from those that are *not in* the interval without making any reference to the endpoints? (We're looking for the endpoints!) If $a < 1/3$ and $a \in (0,1)$, all numbers between a and $1/3$ are also in $(0,1)$. But if $a < 0$, there are points between a and $1/3$ that are not in $(0,1)$. Whether a point is in a set or not is something we can check. Notice that $0 = \inf\{a < 1/3 : (a, 1/3] \subseteq (0,1)\}$.

For each $x \in S$, let $A_x = \{a : a < x \text{ and } (a, x] \subseteq S\}$. Since S is an open set, $A_x \neq \emptyset$. Now A_x is either bounded below or it is not. If A_x is bounded below, let $a_x = \inf A_x$, otherwise let $a_x = -\infty$ (this is acceptable only because a_x is to be an endpoint of an interval). Likewise, we let $B_x = \{b : b > x \text{ and } [x, b) \subseteq S\}$, and $b_x = \sup B_x$ or $+\infty$, as appropriate. The collection $\{(a_x, b_x)\}$ is essentially what we're looking for. We show first that $S = \bigcup_{x\in S}(a_x, b_x)$. Note that this is a set-equality problem.

Let $x \in S$. Since S is open, there is an $\varepsilon > 0$ with $(x-\varepsilon, x+\varepsilon) \subseteq S$. In particular, $(x-\varepsilon, x] \subseteq S$, so that $a_x \le x - \varepsilon < x$ (since $x - \varepsilon$ is an element of the set of which a_x is the infimum). Similarly, we find that $x < x + \varepsilon \le b_x$. Thus $x \in (a_x, b_x)$, and $S \subseteq \bigcup_{x\in S}(a_x, b_x)$. Now suppose $y \in \bigcup_{x\in S}(a_x, b_x)$. There is an $x_0 \in S$ so that $y \in (a_{x_0}, b_{x_0})$. We may assume $a_{x_0} < y < x_0$. Since $y > a_{x_0} = \inf A_{x_0}$, there is an $a \in A_{x_0}$ with $y > a$. This means $y \in (a, x_0] \subseteq S$, so $y \in S$. Thus $\bigcup_{x\in S}(a_x, b_x) \subseteq S$.

Are we done? Not by a long shot. If $S = (0,1)$, we have $(a_x, b_x) = (0,1)$ for *every* $x \in S$. The intervals (a_x, b_x) are not mutually disjoint, and there are uncountably many of them. There is hope, though, because the same interval appears infinitely many times. We will show that this must happen, in the sense that any two of these intervals are either disjoint or are the same. We first show $a_x \notin S$ for any x. This is clear if $a_x = -\infty$, and so we may assume $a_x \in \mathbf{R}$. If $a_x \in S$, there is an $\varepsilon > 0$ with $(a_x - \varepsilon, a_x + \varepsilon) \subseteq S$. Since $a_x = \inf A_x$, there is a $y \in A_x$ with $a_x < y < a_x + \varepsilon$. Then $(a_x - \varepsilon, x] = (a_x - \varepsilon, a_x + \varepsilon) \cup (y, x] \subseteq S$, and so $a_x - \varepsilon \in A_x$, contradicting the way a_x was chosen. Similarly, $b_x \notin S$. Since $S = \bigcup_{x\in S}(a_x, b_x)$, none of the points a_x or b_x are contained in (a_z, b_z) for any z. It follows from Exercise 4.9.2 that any two of these intervals are either disjoint or identical. We will agree to list each interval of $\{(a_x, b_x)\}$ only once.

We have now shown that S can be written as a union of mutually disjoint open intervals. It remains to show that there are only countably many of them. We will put $\{(a_x, b_x)\}$ into one-to-one correspondence with a set of rational numbers. Associate with each interval (a_x, b_x)

that rational number, say q_x, it is guaranteed by the Density theorem to contain.[1] If $(a_x, b_x) \neq (a_y, b_y)$, we have $(a_x, b_x) \cap (a_y, b_y) = \emptyset$, so that $q_x \neq q_y$. The association between the intervals (a_x, b_x) and the numbers q_x is thus one-to-one (it is certainly not onto, but this doesn't matter). Since $\{q_x\} \subseteq \mathbf{Q}$, the set $\{q_x\}$ is countable, and consequently so is the collection $\{(a_x, b_x)\}$. ∎

The final step of this proof can be modified to show that any collection of mutually disjoint open sets in the real line is countable. We can sometimes use Theorem 8.11 to establish results for open *sets* by first proving them for open *intervals*, which is often an easier task (see Exercise 8.4.3).

EXERCISES 8.4

1. (a) Show that the sets A_x and B_x in the proof of Theorem 8.11 are not empty.

   (b) The third paragraph of the proof of Theorem 8.11 would have gone more quickly if we had said "since $a_{x_0} < y < x_0$, $y \in S$." But this would not be correct. Why?

2. (a) If the set S in Theorem 8.11 is unbounded, is the collection of open intervals produced necessarily infinite?

   (b) If the set S in Theorem 8.11 is *bounded*, is the collection of open intervals produced necessarily finite?

3. (a) Show that any open interval is a countable union of *closed* intervals (they need not be disjoint).

   (b) Show that any open set is a countable union of closed intervals.

   (c) Is it possible to represent an open interval as a union of countably many *disjoint* closed intervals?

   (d) What if we allow the collection of closed intervals in (c) to be uncountable? (Look for any *easy* answer.)

   (e) What if we allow the collection of closed intervals in (c) to be uncountable, but require that each of them have positive length? (See Exercise 13.2.4 *after* you have considered this.)

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[1] The Density theorem guarantees the existence of such a rational number, but the nonconstructive nature of this statement may be disturbing. We may proceed this way: The rational numbers can be enumerated: $q_1, q_2, \ldots$. Match the interval (a_x, b_x) with the first number *in this list* that it contains.

4. (a) Adjust the end of the proof of Theorem 8.11 to show that any collection of mutually disjoint open subsets of the real line is countable.

   (b) Show that this is not true for closed sets.

5. Suppose that $(X, \mathcal{T})$ is a topological space having a countable, dense subset. Show that any collection of mutually disjoint, open subsets of X is countable. (The meaning of "dense" in this context is defined in Exercise 8.2.2.)

6. A topology is called **first countable** if it has the following property: For each point x in the topological space, there is a countable collection of open sets $\{U_n^x\}$, each containing x and such that if U is any open set containing x, there is an n so that $U_n^x \subseteq U$. The collection $\{U_n^x\}$ is called a **local neighborhood base at x**. Notice that the collection $\{U_n^x\}$ probably changes from point to point. A topology is called **second countable** if there is a single countable collection $\{U_n\}$ such that, if x is in the space and U is an open set containing x, then there is an n so that $x \in U_n \subseteq U$. Such a collection $\{U_n\}$ is called a **base** (or **basis**) for the topology. Note that a basis need not be countable.

   (a) Show that the standard topology on the real line is first countable.

   (b) Show that the standard topology on the real line is second countable.

   (c) Show that any second countable topology is also first countable.

   (d) Write the definitions of first and second countable in standard symbolic form.

   (e) Write the negations of these two definitions in standard symbolic form.

   (f) Say (in words) what it means for a topological space to be not first countable and to be not second countable.

   (g) If $\{U_\alpha\}$ is a basis for a topology $\mathcal{T}$, show that every element of $\mathcal{T}$ can be written as a union of sets U_α.

   (h) If $(X, \mathcal{T})$ is a topological space and $\{V_\alpha\}$ is a collection of subsets of X with the property that every element of $\mathcal{T}$ can be written as a union of sets, each of which is the intersection of finitely many of the sets V_α, then V_α is called a **subbasis** for $\mathcal{T}$. Show that the collection of open rays is a subbasis for the standard topology on the real line.

7. This is more a project than an exercise. We will build a very strange object called the **Cantor set**. Let $C_0 = [0, 1]$, and $S_1 = (1/3, 2/3)$ (we refer to S_1 as the "open middle third" of C_0). Let $C_1 = C_0 \backslash S_1$. Then C_1 consists of two closed intervals. Now each of these intervals

also has an open middle third. Let $S_2 = (1/9, 2/9) \cup (7/9, 8/9)$—the two open middle thirds of C_1 —and let $C_2 = C_1 \setminus S_2$. Then C_2 consists of *four* closed intervals. Remove the four open middle thirds of C_2 to obtain C_3, and so on. Note that $C_n \supseteq C_{n+1}$ for all n. Let $C = \bigcap_n C_n$.

(a) Show that C is closed.

(b) Show that if $x, y \in C$ and $x < y$, there is a number $z \notin C$ with $x < z < y$. (We say "C contains no nontrivial interval." This and part (a) show that the analogue of Theorem 8.11 for closed sets fails in a big way.)

(c) Compute the sum of the lengths of the intervals removed from C_0 in the construction of C. (You have to remember some calculus to do this. Hint: The answer is 1.)

(d) What is the length of C? We don't know a precise definition of "length" yet, but it might be reasonable to assume that the length of C is: 1− (the sum of the lengths of the intervals removed to make C).

(e). Show that C consists of all elements of C_0 whose ternary expansion contains no 1s (see Exercise 6.3.1).

(f) Show that C is uncountable. [This and part (d) show that any connection between length and cardinality is a mystery.]

(g) Clearly the endpoints of the intervals making up the C_n's are elements of C. But the set of these endpoints is countable (why?), and so it can't be all of C. Find *one* element of C that is not one of these endpoints.

(h) Repeat this construction, but remove "open middle fourths" (or fifths, or whatever you wish). State and prove analogues of each part of the problem for this set.

(i) Modify this construction so that the length of the resulting set is not 0. (You will need to remove pieces at each stage that are different fractions of the length of the intervals remaining in the set.) This is called a "fat" Cantor set. It provides examples in many areas of analysis. How many of the other results in this exercise do hold for your fat Cantor set? (Compare this with Exercise 6.3.1.c.)

(j) Construct a one-to-one correspondence between C and C_0. (Note that elements of C have ternary expansions like $0.2202220\ldots$, while elements of C_0 have *binary* expansions like $0.1101110\ldots$.)

(k) Since C is closed, it contains all its cluster points (that is, $C' \subseteq C$). Show that every point of C is a cluster point of C (so $C' = C$). Such a set is called **perfect**.

(l) Show that every nonempty perfect set is uncountable.

8.5 FUNCTIONS—DIRECT AND INVERSE IMAGES

One important reason for thinking about topology is its usefulness in the study of functions. Before we get to the main issue—continuity—we must learn to look at functions from just the right point of view. We usually think of functions in terms of plugging in x and getting out y. Can we plug a *set* into a function? If $f(x) = x^2$, does $f(\{-1, 0, 1, 2\})$ mean anything? One way we might interpret this is simply to plug the elements of the set into the function one at a time. In this case, $f(\{-1, 0, 1, 2\}) = \{0, 1, 4\}$.

DEFINITION 8.12: Let $f : A \to B$, and $S \subseteq A$. The **direct image of S under f** is given by $f(S) = \{y \in B : \exists x \in S \ni (y = f(x))\}$.

This is illustrated in the following diagram. The set of images, under the function f, of the elements in the shaded area on the left might be the shaded region on the right.

If $f : A \to B$, the direct image of the entire domain, $f(A)$, is sometimes called the **range** of f. This might cause some confusion since when we have also referred to B as the range of f in this situation, even if not every element of B is an output of f. To avoid using the same word for two (possibly different) things, the set B in the expression $f : A \to B$ is sometimes called the **codomain** of f. We will write $f(A)$ for the range of f whenever it is important to distinguish it from the codomain (which will not be very often). Observe that any function $f : A \to f(A)$ is onto; in fact, "$B = f(A)$" may be taken as the definition of "$f : A \to B$ is onto."

If $f : A \to B$ is a one-to-one correspondence, it has an **inverse function**, denoted f^{-1}, whose domain is B and whose range is A and which is defined by: $x = f^{-1}(y) \Leftrightarrow y = f(x)$. This is the same as saying $f(f^{-1}(y)) = y$ and $f^{-1}(f(x)) = x$. (We often use y for elements of B and x for elements of A, but this is only a convenience. We can tell from context whether an element being inserted into a function is in A or B.) If f is one-to-one, it is a one-to-one correspondence between A and $f(A)$,

and we can always define an inverse function $f^{-1} : f(A) \to A$. Now if $f : A \to B$ is one-to-one and $S \subseteq f(A) \subseteq B$, the direct image of S under f^{-1} is given by

$$f^{-1}(S) = \{x \in A : x = f^{-1}(y) \text{ for some } y \in S\}.$$

If we replace "$x = f^{-1}(y)$" with "$y = f(x)$" we find

$$f^{-1}(S) = \{x \in A : y = f(x) \text{ for some } y \in S\}.$$

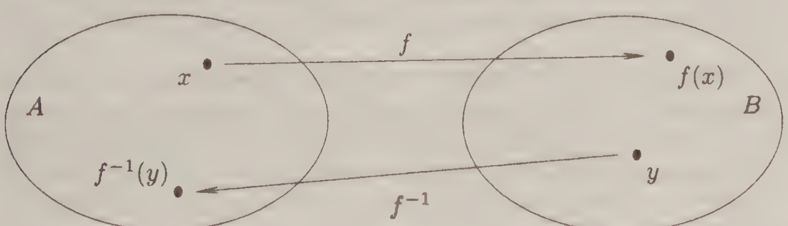
Now "$y = f(x)$ for some $y \in S$" may be abbreviated "$f(x) \in S$," and so

$$f^{-1}(S) = \{x \in A : f(x) \in S\}.$$

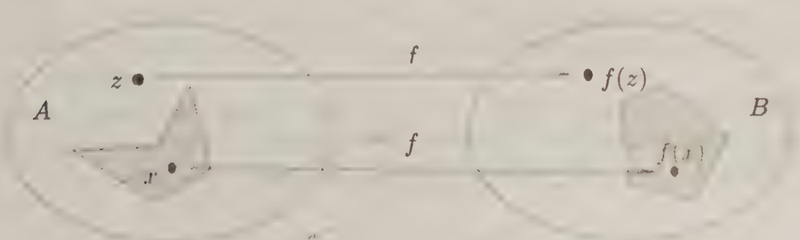
Not only is the last version of this statement easier to look at than the original, we don't have to evaluate an inverse function to use it. In fact, the last formulation makes sense even if f doesn't have an inverse function (that is, if f is not one-to-one) and even if S is not a subset of $f(A)$. Remember that both of these conditions must hold before we can even *consider* the direct image of S under f^{-1}.

DEFINITION 8.13: Let $f : A \to B$ and $S \subseteq B$. The **inverse image of S under f** is given by $f^{-1}(S) = \{x \in A : f(x) \in S\}$.

This definition is best remembered: $x \in f^{-1}(S) \Leftrightarrow f(x) \in S$. The diagram below illustrates the *inverse function* of f:



While the next diagram illustrates the construction of an *inverse image*.



In the second diagram x is an element of the inverse image of the shaded region on the right, while z is not. The most important thing to note here is that this diagram resembles the one describing the *direct image* more than it does the one describing the inverse function (in particular, both the arrows point to the right). The inverse *function* doesn't appear in the picture at all.

Inverse images can *always* be found, and so they are much more useful to us than direct images under inverse functions. But now we are using the notation $f^{-1}(S)$ to stand for two slightly different things: The *direct* image of S under the function f^{-1} and the *inverse* image of S under the function f. You will show in Exercise 8.5.2 that this confusion, though real, is not as dangerous as it might seem. It does, however, influence how we do proofs. *The notation $f^{-1}(S)$ will always mean the inverse image of S under f.* Unless we say otherwise, we *never* assume that a function has an inverse.

EXAMPLES 8.5: 1. Let $f(x) = x^2$ and $A = \{0, 4\}$. Then $f^{-1}(A) = \{0, -2, 2\}$ and $f^{-1}(\{-1\}) = \emptyset$. Note that the inverse image of a set need not have the same cardinality as the set.

2. For any function $f : A \to B$, we have $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(B) = A$. This is different from saying $B = f(A)$, which is not always true. We can't simply apply f to both sides of such an equality since it is not always the case that $f(f^{-1}(S)) = S$. You will examine this issue in Exercise 8.5.4.

3. Let $f(x) = \sin(x)$. Then $f^{-1}(0)$ is *undefined* since f is not one-to-one and so has no inverse function. On the other hand, $f^{-1}(\{0\}) = \{0, \pm\pi, \pm 2\pi, \ldots\}$. (Remember that 0 and $\{0\}$ are very different things.)

We have found two new ways of manipulating sets (direct and inverse images), and we should consider the relationships between these processes and the usual set operations. We will establish two results here. This theorem suggests that inverse images are, in a sense, "better behaved" than direct images, a comforting thought when first encountering a new concept.

THEOREM 8.14: *Suppose $f : A \to B$.*

(a) *If $C, D \subseteq A$, then $f(C \cap D) \subseteq f(C) \cap f(D)$.*
(b) *If $E, F \subseteq B$, then $f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$.*

PROOF: (a) Let $y \in f(C \cap D)$. By the definition of direct image, there is an element x of $C \cap D$ with $y = f(x)$. Then $x \in C$ and $x \in D$. It follows that $y \in f(C)$ and $y \in f(D)$, and so $y \in f(C) \cap f(D)$.

(b) There are two containment proofs to be done. We will use the forward-backward method to show that $f^{-1}(E \cap F) \subseteq f^{-1}(E) \cap f^{-1}(F)$.

▽▽▽

Let $x \in f^{-1}(E \cap F)$.

⋆ ⋆ ⋆

Then $x \in f^{-1}(E) \cap f^{-1}(F)$.

△△△

We insert definitions:

▽▽▽

Let $x \in f^{-1}(E \cap F)$.

$\hookrightarrow$ Then $f(x) \in E \cap F$.

⋆ ⋆ ⋆

$\hookrightarrow$ $x \in f^{-1}(E)$ and $x \in f^{-1}(F)$.

Then $x \in f^{-1}(E) \cap f^{-1}(F)$.

△△△

Another definition, and we're done:

▽▽▽

Let $x \in f^{-1}(E \cap F)$.

Then $f(x) \in E \cap F$.

$\hookrightarrow$ $f(x) \in E$ and $f(x) \in F$.

$x \in f^{-1}(E)$ and $x \in f^{-1}(F)$.

Then $x \in f^{-1}(E) \cap f^{-1}(F)$.

△△△

We may finish the proof with another, similar set argument, or we may observe that each connective in the above argument is "if and only if" (since each step is a definition). This means that the entire argument can simply be reversed (it is *very* rare that we can do one part of an if-and-only-if proof by simply reversing the other part, and we can *never* assume it will work). ∎

The relationship between inverse and direct images and algebraic operations on functions is very subtle. Composition, however, works as we would expect.

THEOREM 8.15: *If $f : A \to B$, $g : B \to C$, and $S \subseteq C$, then*
$$(g \circ f)^{-1}(S) = f^{-1}(g^{-1}(S)).$$

PROOF: We will do half of this proof, leaving the rest as Exercise 8.5.6. This is a good example of how looking at just the right level of detail can make a problem easier. Let $x \in (g \circ f)^{-1}(S)$. By the definition of composition, $(g \circ f)(x) = g(f(x)) \in S$. Since $g(\text{something}) \in S$, it must be that $(\text{something}) \in g^{-1}(S)$, that is, $f(x) \in g^{-1}(S)$. Now we have $f(\text{something else}) \in (\text{some set})$, and so it must be that $(\text{something else}) \in f^{-1}(\text{some set})$. That is, $x \in f^{-1}(g^{-1}(S))$, as desired. ∎

We have consistently used the observation "$x \in f^{-1}(S) \Leftrightarrow f(x) \in S$" but *no* references to inverse functions. Unless we know that f has an inverse function (and we almost never will), we *can't* write expressions like $f^{-1}(y)$, where y is an element of the range of f. This is our final warning on the subject.

EXERCISES 8.5

1. Verify that the definition of inverse function "...is the same as saying $f(f^{-1}(y)) = y$ and $f^{-1}(f(x)) = x$."

2. If f is one-to-one and onto, show that the two interpretations of the expression "$f^{-1}(S)$" are the same. First give them different names, say $U = \{x : x = f^{-1}(y) \text{ for some } y \in S\}$ (the direct image under f^{-1}) and $V = \{x : f(x) \in S\}$ (the inverse image under f). Now show that $U = V$.

3. (a) If $f : A \to B$ is one-to-one and onto, show that f^{-1} is a function.

   (b) Show that the given definition of f^{-1} *doesn't* yield a function if f is not both one-to-one and onto.

4. Show: (a) $f(C \cup D) = f(C) \cup f(D)$.

   (b) $f(C) \backslash f(D) \subseteq f(C \backslash D)$.

   (c) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.

   (d) $f^{-1}(C \backslash D) = f^{-1}(C) \backslash f^{-1}(D)$. (Note again that inverse images are better behaved than direct images.)

   (e) The containment in (b) might be strict (that is, the sets might not be equal).

   (f) The containment in Theorem 8.14.a might be strict.

   (g) $f(f^{-1}(C)) \subseteq C$, and this containment might be strict.

   (h) $C \subseteq f^{-1}(f(C))$, and this containment might be strict.

   (i) If f is one-to-one, show that the sets in (b) are equal.

   (j) If the sets in (b) are always equal, is f necessarily one-to-one?

(k) Are there conditions that can be put on the functions or sets in parts (g) and (h) of this problem and in Theorem 8.14.a that will guarantee that the sets involved are equal?

5. (a) Examine the direct and inverse images of open and closed intervals under the functions $f(x) = x^3$ and $g(x) = x^2$.

   (b) Repeat part (a) for functions x^p and x^q, where p is even and q is odd.

   (c) Repeat part (a) for the function $f(x) = x^2 + x$ or any simple polynomial of your choosing. Develop a conjecture about inverse images of intervals under polynomials.

6. Complete the proof of Theorem 8.15.

7. (a) Suppose $f : A \to B$ and that $S \subseteq f(A)$. Show that the cardinality of $f^{-1}(S)$ is not less than the cardinality of S.

   (b) Show that a function is one-to-one if and only if it has the property that the inverse image of any set with one element has at most one element.

   (c) Suppose a function has the property that the inverse image of any set with *two* elements is either empty or has precisely two elements. Is such a function one-to-one?

   (d) Suppose a function has the property that the inverse image of any countable set is countable and the inverse image of any uncountable set is uncountable. Is such a function one-to-one?

8.6 CONTINUOUS FUNCTIONS

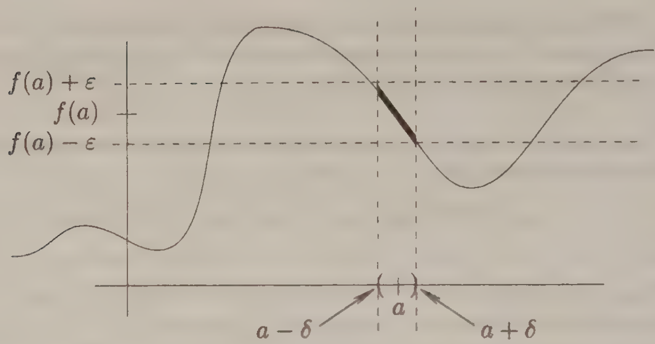
The ε-δ definition of continuity is familiar from calculus:[2]

f is continuous at a if for every $\varepsilon > 0$, there is a $\delta > 0$ so that $|f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta$.

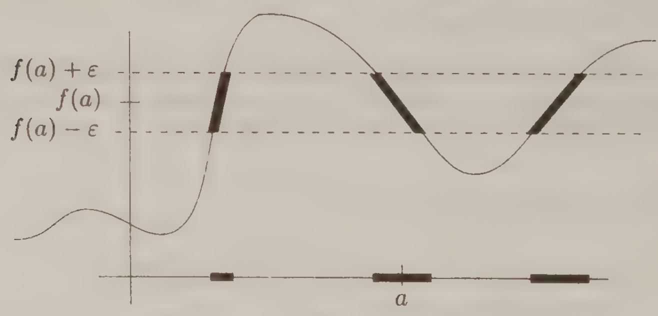
Here is a picture that illustrates the ε-δ definition of continuity:

---

[2] The domain of f must be a neighborhood of a for this to make sense. If we wish to eliminate this requirement, we could change "$|x - a| < \delta$" to "$|x - a| < \delta$ and x is in the domain of f." We will deal with this problem more gracefully later in the chapter. For now we will just assume that the domain of f is a neighborhood of a when it is convenient to do so.



We wish to find a value of δ so that the portion of the graph above the interval $(a-\delta, a+\delta)$ lies between the two dashed horizontal lines. If this is possible for every $\varepsilon > 0$, the function is continuous at a. It would be in keeping with the approach we have adopted to have a purely topological characterization of continuity. Expressions like "$|x-a| < \delta$," with its reference to distance (and implicit reference to ordering) are the sort of thing we try to avoid by looking at things topologically. Observe what happens if we consider *all* the points on the graph that lie between the two dashed horizontal lines, and all points on the x-axis that are sent there by f:



The definition of continuity requires there to be an interval $(a-\delta, a+\delta)$ contained in the set we have constructed on the x-axis. This means that this set must be a neighborhood of a. Notice, too, that the set we have constructed on the x-axis is just the inverse image of $(f(a)-\varepsilon, f(a)+\varepsilon)$ under f. Moreover, if we had begun this process with a *neighborhood* of $f(a)$ (rather than an ε-interval), we would have arrived at the same conclusion. Such a neighborhood would *contain* an ε-interval, and the inverse image of that neighborhood would in turn contain the inverse image of the ε-interval. Putting all of this together, we see that the

definition of continuity of a function f at a point a may be phrased like this:

f is continuous at a if $f^{-1}(U)$ is a neighborhood of a whenever U is a neighborhood of $f(a)$.

If the function f is continuous everywhere (for the time being, we are interested only in functions whose domains are the whole real line), this condition must hold for *all* a. In other words, if we begin with a set on the y-axis that is a neighborhood of each of its points, its inverse image under f must also be a neighborhood of each of its points.

DEFINITION 8.16: A function $f : \mathbf{R} \to \mathbf{R}$ is **continuous** if $f^{-1}(S)$ is open whenever S is open. (We say "Inverse images of open sets are open.")

This definition fulfills our desire for something "purely topological," but the property being defined here is not quite the same as the one in the ε-δ definition. (Be sure you see the difference.) We will reconcile the two definitions in Theorem 8.18. To waylay suspicions that we are being led down a garden path, we will do a short, simple proof. You are invited to supply the ε-δ version.

THEOREM 8.17: *Compositions of continuous functions are continuous.*

PROOF: Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ both be continuous. Consider the inverse image of an open set, say S, under the composition $g \circ f$. By Theorem 8.15, $(g \circ f)^{-1}(S) = f^{-1}(g^{-1}(S))$. Since g is continuous, $g^{-1}(S)$ is open. Since f is continuous, $f^{-1}(g^{-1}(S))$ is open. ∎

The brevity of this proof goes a long way toward explaining why we look at continuity the way we do. There are negative aspects to this. The proof that the *sum* of two continuous functions is continuous is not so easy in these terms. An ε-δ proof is more appropriate in that case.

The use of "whenever" as a quantifier might obscure the logical structure of the definition of continuity (in either form). Some care about this now will make things easier for us later. "Whenever" is a universal quantifier, and the object quantified is the set S. We may write the definition:

$$\forall S \subseteq \mathbf{R}(S \text{ is open} \Rightarrow f^{-1}(S) \text{ is open}).$$

This makes the definition easy to negate. A function is *not* continuous if

$$\exists S \subseteq \mathbf{R} \ni (S \text{ is open and } f^{-1}(S) \text{ is not open})$$

The problem of showing that a function is not continuous is one of finding an example of a *set* with certain properties.

EXAMPLES 8.6: 1. Let $f : \mathbf{R} \to \mathbf{R}$ be constant, say $f(x) = c$ for all x. Then f is continuous: Let S be an open subset of $\mathbf{R}$. If $c \in S$, then $f^{-1}(S) = \mathbf{R}$. On the other hand, if $c \notin S$, then $f^{-1}(S) = \emptyset$. In either case, $f^{-1}(S)$ is open, and so f is continuous. (Note how neatly the definition of a topology fits with this proof.)

2. Let $f(x) = x$. For any set S, $f^{-1}(S) = S$, if S is open (considered as a subset of the range) then S is open (considered as a subset of the domain), and so f is continuous.

3. Let $f(x) = 1$ if $x > 0$ and $f(x) = 0$ if $x \leq 0$. As we suspect, f is not continuous. Let $S = (-1/2, 1/2)$, which is open. Then $f^{-1}(S) = (-\infty, 0]$, which is not open.

We will now examine the connection between the ε-δ definition of continuity and our definition. The ε-δ definition describes continuity at a point, while ours deals with the whole real line. Recall that a function is continuous on a set (according to the ε-δ definition) if it is continuous at each point of the set. Let us say that a function is "continuous(ε)" if it satisfies the ε-δ definition at each point of the real line, and "continuous($\mathcal{T}$)" if it satisfies the definition of continuity we have given here.

THEOREM 8.18: *A function $f : \mathbf{R} \to \mathbf{R}$ is continuous(ε) if and only if it is continuous($\mathcal{T}$).*

We will show that if f is continuous(ε) then it is continuous($\mathcal{T}$). The other half of the proof is Exercise 8.6.2. Our proof will test the limits of the forward-backward method. After establishing the result in this way, we will also present a "streamlined" version of the proof distilled from the forward-backward version with all the grace that hindsight can provide. Which is the better proof? The feeling that elegance is the major portion of quality seems to indicate that the second one is much better. This feeling is practically the definition of mathematics. On the other hand, one might argue that the "better" proof is the one we can find ourselves. This is a personal, aesthetic decision.

Before we begin, we observe that the ε-δ definition of "f is continuous at x" may be written in the following way:

f is continuous at x if, for any $\varepsilon > 0$, there is a $\delta > 0$ so that
$$y \in (x - \delta, x + \delta) \Rightarrow f(y) \in (f(x) - \varepsilon, f(x) + \varepsilon).$$

PROOF 1: We wish to establish that f is continuous($\mathcal{T}$). The definition of continuous($\mathcal{T}$) is universally quantified, and so the proof must begin and end like this:

Let S be an open set.

$\star \quad \star \quad \star$

Then $f^{-1}(S)$ is open.

We will do a few backward steps. First, we insert the definition of "open," as it relates to $f^{-1}(S)$:

Let S be an open set.

$\star \quad \star \quad \star$

$\hookrightarrow$ $f^{-1}(S)$ is a neighborhood of each of its points.

Then $f^{-1}(S)$ is open.

The statement we have just inserted is also universally quantified. To prove it, we must select an element of $f^{-1}(S)$ and show that $f^{-1}(S)$ is a neighborhood of that particular element:

Let S be an open set.

$\hookrightarrow$ Let $x \in f^{-1}(S)$.

$\star \quad \star \quad \star$

$\hookrightarrow$ $f^{-1}(S)$ is a neighborhood of x.

$f^{-1}(S)$ is a neighborhood of each of its points.

Then $f^{-1}(S)$ is open.

Now we will make a forward step and a backward step. We may insert the definitions of both "$x \in f^{-1}(S)$" and "$f^{-1}(S)$ is a neighborhood of x." At this point, we will use a bit of foresight. The main tool we have available—the definition of continuous(ε)—will eventually give us a number that measures a distance in the domain of f. That number will be called δ. The set $f^{-1}(S)$ is also in the domain of f. We take this into account when we choose a name for the number we introduce in our backward step.

▽▽

Let S be an open set.

Let $x \in f^{-1}(S)$.

$\hookrightarrow f(x) \in S$.

⋆ ⋆ ⋆

$\hookrightarrow \exists \delta > 0 \ni ((x - \delta, x + \delta) \subseteq f^{-1}(S))$.

$f^{-1}(S)$ is a neighborhood of x.

$f^{-1}(S)$ is a neighborhood of each of its points.

Then $f^{-1}(S)$ is open.

△△

Now we will make a few forward steps. Since S is open, we can add:

▽▽

Let S be an open set.

Let $x \in f^{-1}(S)$.

$f(x) \in S$.

$\hookrightarrow S$ is a neighborhood of $f(x)$.

⋆ ⋆ ⋆

$\exists \delta > 0 \ni ((x - \delta, x + \delta) \subseteq f^{-1}(S))$.

$f^{-1}(S)$ is a neighborhood of x.

$f^{-1}(S)$ is a neighborhood of each of its points.

Then $f^{-1}(S)$ is open.

△△

...and the definition of neighborhood [again we think of continuous(ε), which will require a distance called ε in the range of the function] ...

▽▽

Let S be an open set.

Let $x \in f^{-1}(S)$.

$f(x) \in S$.

S is a neighborhood of $f(x)$.

$\hookrightarrow \exists \varepsilon > 0 \ni ((f(x) - \varepsilon, f(x) + \varepsilon) \subseteq S)$.

⋆ ⋆ ⋆

$\exists \delta > 0 \ni ((x - \delta, x + \delta) \subseteq f^{-1}(S))$.

$f^{-1}(S)$ is a neighborhood of x.

$f^{-1}(S)$ is a neighborhood of each of its points.

Then $f^{-1}(S)$ is open.

△△

We insert the definition of "subset":

▽▽

Let S be an open set.

Let $x \in f^{-1}(S)$.

$f(x) \in S$.

S is a neighborhood of $f(x)$.

$\exists \varepsilon > 0 \ni ((f(x) - \varepsilon, f(x) + \varepsilon) \subseteq S)$.

$\longleftrightarrow \exists \varepsilon > 0 \ni \forall z (z \in (f(x) - \varepsilon, f(x) + \varepsilon) \Rightarrow z \in S)$.

⋆ ⋆ ⋆

$\exists \delta > 0 \ni ((x - \delta, x + \delta) \subseteq f^{-1}(S))$.

$f^{-1}(S)$ is a neighborhood of x.

$f^{-1}(S)$ is a neighborhood of each of its points.

Then $f^{-1}(S)$ is open.

△△

Now we come to the hardest step of the proof. Consider the last line we have reached by forward steps and look back at the statement of the ε-δ definition just before the proof. Using this definition, we can force $f(y)$ to be an element of $(f(x) - \varepsilon, f(x) + \varepsilon)$ by putting y in just the right place [and such an $f(y)$ would be one of the z's in the last forward statement in the proof]. Since this can be done for *any* ε, it can be done for the particular one we have at hand.

▽▽

Let S be an open set.

Let $x \in f^{-1}(S)$.

$f(x) \in S$.

S is a neighborhood of $f(x)$.

$\exists \varepsilon > 0 \ni ((f(x) - \varepsilon, f(x) + \varepsilon) \subseteq S)$.

$\exists \varepsilon > 0 \ni \forall z (z \in (f(x) - \varepsilon, f(x) + \varepsilon) \Rightarrow z \in S)$.

$\longleftrightarrow \exists \delta > 0 \ni (y \in (x - \delta, x + \delta) \Rightarrow f(y) \in (f(x) - \varepsilon, f(x) + \varepsilon))$.

⋆ ⋆ ⋆

$\exists \delta > 0 \ni ((x - \delta, x + \delta) \subseteq f^{-1}(S))$.

$f^{-1}(S)$ is a neighborhood of x.

$f^{-1}(S)$ is a neighborhood of each of its points.

Then $f^{-1}(S)$ is open.

△△

Combining the statement we have just inserted with the line above it, we have

▽▽▽

Let S be an open set.

Let $x \in f^{-1}(S)$.

$f(x) \in S$.

S is a neighborhood of $f(x)$.

$\exists \varepsilon > 0 \ni ((f(x) - \varepsilon, f(x) + \varepsilon) \subseteq S)$.

$\exists \varepsilon > 0 \ni \forall z (z \in (f(x) - \varepsilon, f(x) + \varepsilon) \Rightarrow z \in S)$.

$\exists \delta > 0 \ni (y \in (x - \delta, x + \delta) \Rightarrow f(y) \in (f(x) - \varepsilon, f(x) + \varepsilon))$.

$\hookrightarrow \exists \delta > 0 \ni (y \in (x - \delta, x + \delta) \Rightarrow f(y) \in S)$.

★ ★ ★

$\exists \delta > 0 \ni ((x - \delta, x + \delta) \subseteq f^{-1}(S))$.

$f^{-1}(S)$ is a neighborhood of x.

$f^{-1}(S)$ is a neighborhood of each of its points.

Then $f^{-1}(S)$ is open.

△△△

Inserting the definition of inverse image and noting that the line so formed is the definition of "subset," we are done:

▽▽▽

Let S be an open set.

Let $x \in f^{-1}(S)$.

$f(x) \in S$.

S is a neighborhood of $f(x)$.

$\exists \varepsilon > 0 \ni ((f(x) - \varepsilon, f(x) + \varepsilon) \subseteq S)$.

$\exists \varepsilon > 0 \ni \forall z (z \in (f(x) - \varepsilon, f(x) + \varepsilon) \Rightarrow z \in S)$.

$\exists \delta > 0 \ni (y \in (x - \delta, x + \delta) \Rightarrow f(y) \in (f(x) - \varepsilon, f(x) + \varepsilon))$.

$\exists \delta > 0 \ni (y \in (x - \delta, x + \delta) \Rightarrow f(y) \in S)$.

$\hookrightarrow \exists \delta > 0 \ni (y \in (x - \delta, x + \delta) \Rightarrow y \in f^{-1}(S))$.

$\exists \delta > 0 \ni ((x - \delta, x + \delta) \subseteq f^{-1}(S))$.

$f^{-1}(S)$ is a neighborhood of x.

$f^{-1}(S)$ is a neighborhood of each of its points.

Then $f^{-1}(S)$ is open. ∎

△△△

As always, the forward-backward method did not let us avoid the difficult steps in the proof, but it made clear where they occurred. This allowed us to focus our attention on them, and only when it was necessary. Here is the streamlined version of the proof. You should look back and forth between it and the forward-backward version to see where the steps come from and why they fit together as they do (some of the connections are pretty subtle).

PROOF 2: Suppose that f is continuous(ε). Let S be an open set. Let $x \in f^{-1}(S)$ and let $y = f(x)$. Since $y \in S$ and S is open, there is an $\varepsilon > 0$ so that $(y-\varepsilon, y+\varepsilon) \subseteq S$. Since f is continuous(ε), there is a $\delta > 0$, so that $f((x-\delta, x+\delta)) \subseteq (y-\varepsilon, y+\varepsilon)$. For each $z \in (x-\delta, x+\delta)$, we have $f(z) \in (y-\varepsilon, y+\varepsilon) \subseteq S$. This means $(x-\delta, x+\delta) \subseteq f^{-1}(S)$, and so $f^{-1}(S)$ is a neighborhood of x, $f^{-1}(S)$ is open, and f is continuous($\mathcal{T}$). ∎

EXERCISES 8.6

1. Give an ε-δ proof of Theorem 8.17.

2. Complete the proof of Theorem 8.18.

3. (a) If f is continuous, show that inverse images of closed sets are closed.

   (b) Go back and review your answer to Exercise 8.5.5.

4. Prove or disprove: The function f is continuous(ε) at the point a if and only if there is a neighborhood U of a such that f is continuous($\mathcal{T}$) on U.

5. Show that the function f is continuous at the point a if and only if, for every $\varepsilon > 0$, there is a $\delta > 0$ so that $|f(x) - f(y)| < \varepsilon$ whenever x and y are both in the interval $(a-\delta, a+\delta)$.

6. (a) Show that if X has the discrete topology, *every* function whose domain is X is continuous (regardless of the topology on the codomain). The discrete topology was defined in Exercise 8.2.3.

   (b) Show that if Y has the indiscrete topology, every function whose codomain is Y is continuous (regardless of the topology on the domain)

   (c) Show that constant functions are always continuous.

   (d) Show that if X has the indiscrete topology and Y does not, the *only* functions $f : X \to Y$ that are continuous are constants.

(e) Show that the identity function ($f(x) = x$) from a topological space to itself is always continuous.

(f) Suppose X_1 and X_2 are equal as sets but have different topologies (and so they are not equal as topological spaces). Show that the identity function $f : X_1 \to X_2$ need not be continuous.

(g) What conditions on the topologies on X_1 and X_2 in part (f) would guarantee that the identity function is continuous?

8.7 RELATIVE TOPOLOGIES

Our definition of continuity does not tell us what it means for a function to be continuous if its domain is not the whole real line. Just about all of the important theorems of calculus concern functions whose domains are closed, bounded intervals, and so it is particularly important that we know what it means for such functions to be continuous. In view of Definition 8.16, all we really need is to define what it means for a subset of an interval to be open. Be warned that *the following is not standard terminology.*

DEFINITION 8.19: If $T \subseteq S \subseteq \mathbf{R}$, we say T is ***open in** S if there is an open subset U of $\mathbf{R}$ so that $T = S \cap U$.

This should be pronounced "star-open." For now, we reserve "open" to mean "an open subset of the real line." With a little practice we will be able to distinguish the difference from context and will just say "open" in both cases. Notice that this definition applies to *any* subset of the real numbers, not just intervals. The collection of *open subsets of a set S is called the **relative topology on** S or the topology S **inherits** from $\mathbf{R}$. You will show in Exercise 8.7.1 that a relative topology is indeed a topology. The definition of continuity for a function defined on a subset of the real line is now quite natural:

DEFINITION 8.20: If $S, T \subseteq \mathbf{R}$, a function $f : S \to T$ is **continuous** if the inverse image of any *open set in T is a *open set in S.

If $(X, \mathcal{T})$ is any topological space, we usually refer to the elements of $\mathcal{T}$ as "open." If we follow this convention, this definition (without the $\star$s) makes sense for functions between any two topological spaces.

EXAMPLES 8.7: 1. Each natural number is a *open subset of $\mathbf{N}$. For example, $\{7\} = (6.5, 7.5) \cap \mathbf{N}$. In fact, *every* subset of $\mathbf{N}$ is *open. It follows that *every* function whose domain is $\mathbf{N}$ is continuous. This is true

of any set each of whose points is *open. We may interpret this in terms of pointwise continuity (the ε-δ kind) by saying that if $x \in S$ is a point such that $\{x\}$ is a *open subset of S, then any function $f : S \to \mathbf{R}$ is continuous at x. This is a useful observation. ("$\{x\}$ is a *open subset of S" is equivalent to "x is an isolated point of S"—see Exercise 7.5.9.)

2. Let $S = [0, 1]$. Then $(1/2, 1]$ is *open in S since $(1/2, 1] = (1/2, 3) \cap S$, and $(1/2, 3)$ is open. Note that $(1/2, 1]$ is *not* an open subset of $\mathbf{R}$ and is not a *open subset of $[0, 2]$. Whether $T \subseteq S$ is *open depends on both T and S.

3. If $f : \mathbf{R} \to \mathbf{R}$ is continuous, it is also continuous when its domain is restricted to $[0, 1]$ (see Exercise 8.7.5). In fact, every continuous function on $[0, 1]$ arises in this way. We may just extend such a function to the whole real line by saying $f(x) = f(1)$ if $x > 1$ and $f(x) = f(0)$ if $x < 0$. It is a much deeper result that this same process can be carried out for any function whose domain is a closed, bounded set. This is not true if the domain is an open interval, as the next example shows.

4. Now let $S = (0, 1)$. You will show in Exercise 8.7.2 that a subset of S is *open if and only if it is open when considered as a subset of all of $\mathbf{R}$. However, the function given by $f(x) = 1/x$ is continuous on S, but it can't be obtained as a restriction to S of a continuous function on $\mathbf{R}$. Win some, lose some. Note also that the function given by $f(x) = 0$ for $x < 1$ and $f(x) = 1$ for $x > 1$ is continuous when restricted to $(0, 1)$ but not when restricted to $[0, 1]$.

5. The function in Example 8.6.3 is continuous on $(-\infty, 0]$ and continuous on $(0, \infty)$, but not on $[0, \infty)$.

EXERCISES 8.7

1. If $S \subseteq \mathbf{R}$, show that the *open subsets of S are a topology on S.

2. (a) If S is open and $T \subseteq S$, show that T is *open if and only if T is open.

   (b) Give an example to show that this may not be so if S is not open.

   (c) If $T \subseteq S$ and T is open, show that T is *open no matter what S is.

3. (a) Suppose $Y \subseteq X$ and f is continuous on X. Show that f is continuous on Y.

   (b) Show that f can be continuous on Y but not continuous on X.

(c) If f is continuous on $A \cup B$, it is continuous on A and on B.

(d) This is true for the union of any number of sets.

(e) This is not always true for intersections.

(f) If f is continuous on A and on B, it is continuous on $A \cap B$.

(g) This is not always true for unions.

4. (a) Suppose $x \in S$ is an isolated point of S. Show that $\{x\}$ is open in S.

(b) With S and X as in (a), show that every function defined on S is continuous(ε) at x.

(c) If S is a set with no cluster points, show that every function defined on S is continuous.

5. (a) Suppose $f : [a, b] \to \mathbf{R}$ is continuous. Show that the function g defined by

$$g(x) = \begin{cases} f(a) & x < a \\ f(x) & a \leq x \leq b \\ f(b) & x > b \end{cases}$$

is continuous on the whole real line. (This is called an **extension** of f. While there is only one way to *restrict* a function to a domain smaller than its original one, there are many ways to *extend* one.)

(b) If the domain of the function in (a) is an *open* interval, show that it might not be possible to extend it to a continuous function on the whole real line.

(c) Suppose $f : \mathbf{R} \to \mathbf{R}$ is continuous and that $B = f(\mathbf{R})$. Show that f considered as a function $f : \mathbf{R} \to B$ is continuous.

6. (a) Show that the topology that $\mathbf{N}$ inherits from $\mathbf{R}$ is the same as the discrete topology on $\mathbf{N}$.

(b) If $S \subseteq \mathbf{R}$ and every point of S is an isolated point, show that the topology that S inherits from $\mathbf{R}$ is the discrete topology.

(c) The definition of relative topology applies to any set that is a subset of a topological space. Show that if $S \subseteq X$ and X has the discrete topology, then S inherits the discrete topology, and if X has the indiscrete topology, then S inherits the indiscrete topology.

(d) Suppose $S \subseteq X$, where X is a topological space. Give examples to show that S can inherit the indiscrete topology even if X does not have the indiscrete topology, and that S can inherit the discrete topology even if X does not have the discrete topology.

Chapter 9

Sequences

9.1 AN APPROXIMATION PROBLEM

Here is a process for finding an approximation to $\sqrt{2}$. Let $f(x) = x^2 - 2$. We will make a succession of guesses to the solution of the equation $f(x) = 0$. To find our first guess, we observe that f is continuous, $f(1) < 0$, and $f(2) > 0$. By the Intermediate Value theorem,[1] there is a number a between 1 and 2 with $f(a) = 0$. We now know that our problem has a solution, and roughly where it is. At each stage of this process we will have both a guess for the solution and a current interval in which the solution is known to lie. Let's take $x_1 = 1$ (the left endpoint of the current interval) as our first guess. This is not the solution to the equation, but it is at most one unit away from the solution. (We suspect it is closer than that, but with the information we have so far this is the best we can guarantee.) If we're happy with a guess that could be off by one unit, we're done. But we can do better. At $x = 3/2$ (the center of $[1, 2]$), $f(x)$ is positive. By the Intermediate Value theorem again, we know there is a solution to $f(x) = 0$ between 1 and $3/2$. As before, we may use the left endpoint of our current interval as our estimate, making $x_2 = 1$ also our second guess. We know more about it now, though: We know it's at most $1/2$ unit from the answer.

We can keep this up as long as we like. The center of $[1, 3/2]$ is $5/4$. The function is negative at $5/4$ and positive at $3/2$, and so our next guess is $x_3 = 5/4$, which is off by at most $1/4$. We might hit the solution exactly at some stage, or we might go on indefinitely. If we don't find the exact answer, just what do we get out of this process? We get a collection of guesses: $x_1, x_2, \ldots$, each accompanied by an estimate of how it differs from the answer. Each of these error estimates is smaller than the previous one (even though the difference between the guesses and the answer may not actually decrease with each step). We can say this about our "solution":

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[1] We will prove the Intermediate Value theorem in Chapter 12.

> If we say *beforehand* how much error we can tolerate, there is a stage in this process where we can *guarantee* that the desired accuracy is achieved for that and *all subsequent* estimates.

This is the essence of the theory of sequences. We need only make these observations precise. Notice that in our example we have created a function that takes each natural number n to our guess x_n.

DEFINITION 9.1: A **real sequence** is a function $x : \mathbf{N} \to \mathbf{R}$.

"Real" in this definition refers to the range of the function. One can consider "rational" sequences or "complex" sequences. It is customary to use x, y, or z, rather than f, g, or h, to name sequences, and to denote the value of x at n by x_n rather than $x(n)$ (which would be more correct). If we wish to refer to a whole sequence, we write (x_n) or X. This general abuse of notation shouldn't cause any confusion. It is important, though, to distinguish a *sequence* (x_n) from its *range* $\{x_n\}$. The parentheses denoting the sequence indicate that the order in which the terms appear is important. We can refer to the "third term" of a sequence, but not to the "third element" of a set.

EXERCISES 9.1

1. Show that the solution to the equation in the example that opens the section is never found exactly.

2. (a) The process described above called the **bisection method**. Continue it until you have an approximation to $\sqrt{2}$ that you can guarantee is accurate to two decimal places.

   (b) After the first guess and the first interval are found in the bisection method, one can say exactly how many steps will be necessary before the desired accuracy can be guaranteed. Explain.

   (c) Use the bisection method to find an approximate solution to the equation $\cos x = x$ that is accurate to two decimal places (you will need a calculator).

   (d) Show that the bisection method never results in a guess that is less than the previous one.

   (e) The biscetion method resembles the procedure developed in Chapter 6 for finding the decimal expansion of a number. Discuss the benfits and drawbacks of dividing the working interval into *ten* parts instead of two at each stage of this process.

9.2 CONVERGENCE

We now make precise our comments about the approximating procedure at the beginning of the chapter.

DEFINITION 9.2: (a) A sequence (x_n) is said to **converge to L** if, for every $\varepsilon > 0$, there is a number N_ε so that $|x_n - L| < \varepsilon$ whenever $n > N_\varepsilon$. If this is the case, we say L is the **limit** of (x_n) and write $\lim x_n = L$. (This is usually called the "ε-N version" of the definition.)

(b) A sequence (x_n) is said to **converge** if there is a number L so that (x_n) converges to L. A sequence that does not converge is said to **diverge**.

EXAMPLES 9.2: 1. Let $x_n = 1/n$ and let $\varepsilon > 0$ be given. By Corollary 6.2.b, there is a natural number N_ε so that $1/N_\varepsilon < \varepsilon$. If $n > N_\varepsilon$, we have $|1/n - 0| = 1/n < 1/N_\varepsilon < \varepsilon$. Thus $\lim 1/n = 0$. Note that we had to guess that the limit is 0 in order to begin.

2. In the example at the beginning of the chapter, the difference between x_n and $\sqrt{2}$ is no more than $1/2^n$. Since $n < 2^n$, we have (by choosing N_ε as in the previous example), $|x_n - \sqrt{2}| \leq 1/2^n < 1/n < 1/N_\varepsilon < \varepsilon$, and so $\lim x_n = \sqrt{2}$, as we suspected.

NOTES: 1. L must be a number. A sequence can't converge to ∞.

2. It is reasonable to assume that N_ε depends on ε. If we make our "error tolerance" smaller, we expect to be required go further along in the process to achieve it. You will examine the relationship between ε and N_ε more closely in Exercise 9.2.1. The dependence of N_ε on ε is usually suppressed in notation, and we just write N.

3. We don't write $\lim_{n\to\infty} x_n = L$ simply because the limiting process in a sequence is always the same ($n \to \infty$). It can be useful to think of the problem as a "limit at infinity," though, since we have seen these in calculus.

To bring our study of sequences in line with the rest of our work, we should find a topological characterization of convergence. Note that $|x_n - L| < \varepsilon$ if and only if $x_n \in (L-\varepsilon, L+\varepsilon)$, and that $(L-\varepsilon, L+\varepsilon)$ is a neighborhood of L. This suggests the following:

ALTERNATE DEFINITION 9.3: (x_n) converges to L if, for each neighborhood V of L, there is a number N_V so that $x_n \in V$ whenever $n > N_V$.

The phrase "the sequence does so and so for $n > N$" turns up often enough to deserve a name:

DEFINITION 9.4: (a) A sequence (x_n) is **eventually** in the set S if there is a number N so that $x_n \in S$ whenever $n > N$.

(b) A sequence (x_n) is **frequently** in a set S if for any natural number N, there is an $n > N$ for which $x_n \in S$.

We also may use phrases such as "eventually positive" and "frequently less than 10." It is important to remember that "eventually" and "frequently" now have precise meanings, which may or may not coincide with everyday usage. We may state the definition of convergence more succinctly:

FINAL DEFINITION 9.5: The sequence (x_n) converges to L if it is eventually in any neighborhood of L.

EXAMPLES 9.2: 3. Let $x_n = (-1)^n$. Then (x_n) does not converge. Let z be any real number and $V = (z-1, z+1)$. By Theorem 4.19, if (x_n) is eventually in V, it must eventually have the property that, for any m and n, $|x_n - x_m| < 2$. The sequence (x_n) doesn't have this property, and so it can't converge to any number.

Notice that we could have picked any $\varepsilon \leq 2$ and let $V = (z-\varepsilon/2, z+\varepsilon/2)$. We see that if a sequence does converge to some number z, and $\varepsilon > 0$ is given, it must eventually be the case that $|x_n - x_m| < \varepsilon$ (curiously, the last statement makes no reference to z). This is the basis for the deepest part of the theory of convergent sequences and may be stated roughly like this:

If the terms in a sequence get close to a limit,
they must get close to each other.

We will examine the ramifications of this in the next chapter. The numbers $(-1)^n$ don't get close to each other, and so the sequence can't have a limit.

EXERCISES 9.2

1. (a) Suppose that (x_n), ε and N_ε are as in Definition 9.2 and that $\delta > \varepsilon$. Show that it is also the case that $|x_n - L| < \delta$ whenever $n > N_\varepsilon$.

   (b) Explain why this means that "N_ε gets bigger as ε gets smaller."

   (c) Is the statement in (b) strictly true? In other words, is it really true that $\delta > \varepsilon \Rightarrow N_\delta < N_\varepsilon$?

(d) With (x_n) and ε as in Definition 9.2, let

$$\nu(\varepsilon) = \min\{N : |x_n - L| < \varepsilon \text{ whenever } n > N\}.$$

Show that $\delta > \varepsilon \Rightarrow \nu(\delta) \leq \nu(\varepsilon)$.

2. Give examples of a sequence of *rational* numbers whose limit is *irrational* and a sequence of *irrational* numbers whose limit is *rational.*

3. (a) Interpret the phrase "eventually $\Rightarrow$ frequently" and prove it.

   (b) Show that "frequently" does not imply "eventually."

   (c) Show that a sequence is eventually in a set if there are only finitely many values of n for which it is *not* in the set.

   (d) Show that a sequence is frequently in a set if it is in the set for infinitely many values of n.

4. (a) Suppose $f : \mathbf{R} \to \mathbf{R}$ and that $\lim_{x\to\infty} f(x) = L$ (in the usual calculus sense). Show that $\lim f(n) = L$.

   (b) Show that it is possible to have $\lim f(n) = L$ even if $\lim_{x\to\infty} f(x)$ doesn't exist.

   (c) Give a condition on $f(x)$ that will guarantee that $\lim f(n) = L$ implies $\lim_{x\to\infty} f(x) = L$.

5. (a) Guess the value of $\lim \frac{4n^2}{n^2+1}$ and use the ε-N definition to show that your guess is correct.

   (b) Repeat part (a) for $\lim \frac{3n^2}{n^3+1}$.

6. (a) Show that $\lim x_n = L$ if and only if the sequence described by $x_1, L, x_2, L, x_3, L, \ldots,$ converges.

   (b) Show that $\lim x_n = \lim y_n$ if and only if the sequence described by $x_1, y_1, x_2, y_2, \ldots$ converges.

7. Show that a sequence can't be eventually in both of two disjoint sets.

8. (a) Prove that if there is an $\varepsilon > 0$ so that (x_n) is not eventually in any interval of length ε, then (x_n) diverges.

   (b) Use this to prove (yet again) that $((-1)^n)$ diverges.

   (c) Carefully state the converse of (a) and either prove or disprove it.

9. Given a sequence (x_n), define a sequence (a_n) by $a_n = \frac{x_1+x_2+\cdots+x_n}{n}$

   (a) If $\lim x_n = L$, show that $\lim a_n = L$.

   (b) Give an example of a sequence x_n where $\lim a_n$ exists but $\lim x_n$ does not.

9.3 CONVERGENT SEQUENCES

We now establish some basic results describing the behavior of convergent sequences and the relationship between such sequences and their limits.

THEOREM 9.6: *A sequence can have at most limit.*

PROOF: Suppose $L \neq M$. By Exercise 4.10.4, we can find neighborhoods U of L and V of M with $U \cap V = \emptyset$. Since no sequence can be eventually in two disjoint sets (Exercise 9.2.7), it is impossible for a sequence to converge to both L and M. ∎

The proof of Theorem 9.6, with its reference to neighborhoods, is essentially topological. But Exercise 4.10.4, on which the proof is based, does not hold in every topological space, and neither does Theorem 9.6. You will see how dramatically it can fail in Exercise 9.6.6. We say a sequence (x_n) is **bounded** if its range $\{x_n\}$ is a bounded set.

THEOREM 9.7: *A convergent sequence is bounded.*

PROOF: Suppose (x_n) converges to L and let $\varepsilon = 1$. There is a natural number N so that $x_n \in (L-1, L+1)$ whenever $n > N$. Let $B = \max\{L+1, x_1, x_2, \ldots, x_n\}$. Then $x_n \leq B$ for all n (x_n is in the set of which B is the maximum if $n \leq N$, and is less than $L+1$ if $n > N$). Likewise, $x_n \geq \min\{L-1, x_1, x_2, \ldots, x_n\}$ for all n. ∎

Observe that the converse of this theorem is not true (a bounded sequence need not converge). This proof illustrates an important technique of analysis. We break the problem carefully into cases (even though it might not seem at first to be a proof by cases), and obtain information about each case separately, sometimes in different ways. Here the cases are "$n \leq N$" and "$n > N$." The part of the sequence for which $n > N$ is bounded because the sequence converges, while the part of the sequence for which $n \leq N$ is bounded because it's a finite set.

It seems that if "x_n gets close to L," then the distance between x_n and L must "get close to zero." This translates easily into the following, whose proof is Exercise 9.3.1.

THEOREM 9.8: *Let (x_n) be a sequence, $L \in \mathbf{R}$, and $d_n = |x_n - L|$. Then $\lim x_n = L$ if and only if $\lim d_n = 0$.* ∎

EXERCISES 9.3

1. Prove Theorem 9.8.

2. Show that $\lim x_n = L$ if and only if the following holds: Given $\varepsilon > 0$ and any positive real number b, there is an $N \in \mathbf{N}$ so that $|x_n - L| < b\varepsilon$ whenever $n > N$. Keep a lookout for all the places in the chapter where this could save us some work.

3. (a) Show that $n^2 \geq n$ for all $n \in \mathbf{N}$.

   (b) Show in detail that $\lim \frac{1}{n^2} = 0$.

4. (a) Show that, if $\lim x_n = 0$ and (y_n) is bounded, then $\lim x_n y_n = 0$ (whether (y_n) converges or not).

   (b) Does (a) remain true if $\lim x_n$ exists but is not 0?

   (c) Is is possible to have $x_n y_n$ converge even if y_n is unbounded?

5. (a) Show that if $\lim x_n = L$, then $\lim |x_n| = |L|$.

   (b) Show that the converse of (a) is not true.

   (c) Show that if $\lim |x_n| = 0$, then $\lim x_n = 0$.

   (d) What property of 0 makes (c) true?

6. Suppose X is a topological space that does *not* have the property described in Exercise 4.10.4. Show that there *must* be a sequence in X that converges to two different limits. (A topological space having the property of Exercise 4.10.4 is called a **Hausdorff space.**)

9.4 SEQUENCES AND ORDER

The most important aspects of the interplay between sequences and the order structure of the real line are seen in the behavior of increasing and decreasing sequences. Our examination of this topic must wait until the next chapter, but we can say a few things now about sequences of positive numbers. Observe that the following theorem draws its conclusions about sequences from properties of their individual terms. In the next chapter we will be more concerned with sequences as whole objects.

THEOREM 9.9: (a) *If (x_n) converges and $x_n \geq 0$ for all n, then* $\lim x_n \geq 0$.

(b) *If (x_n) and (y_n) converge and $x_n \leq y_n$ for all n, then* $\lim x_n \leq \lim y_n$.

(c) *If (x_n) and (y_n) both converge to L and $x_n \leq z_n \leq y_n$ for all n, then (z_n) converges and $\lim z_n = L$.*

PROOF: (a) Suppose $a < 0$. Then $(-\infty, 0)$ is a neighborhood of a containing *no* element of $\{x_n\}$, and so (x_n) can't converge to a.

(b) Let $\lim x_n = L$ and $\lim y_n = M$. Suppose $L > M$ and let $\varepsilon = (L-M)/2$. Then (x_n) is eventually in the interval $(L-\varepsilon, L+\varepsilon)$ and (y_n) is eventually in $(M-\varepsilon, M+\varepsilon)$. But every element of $(L-\varepsilon, L+\varepsilon)$ is *larger* than every element of $(M-\varepsilon, M+\varepsilon)$, and so for n sufficiently large, we have $x_n > y_n$, a contradiction.

(c) Let $\varepsilon > 0$ be given. There is a natural number N_1 such that $x_n > L-\varepsilon$ whenever $n > N_1$. Likewise, there is a natural number N_2 such that $y_n < L+\varepsilon$ whenever $n > N_2$. Let $N = \max\{N_1, N_2\}$, so that both inequalities hold for $n > N$. Then if $n > N$, $L-\varepsilon < x_n \le z_n \le y_n < L+\varepsilon$, and so $\lim z_n = L$. ■

Part (c) of Theorem 9.9 is often called the **Squeeze theorem**. Its wording reminds us that whether a sequence converges and the value of its limit are separate pieces of information.

EXAMPLES 9.4: 1. By letting us deal with upper and lower estimates instead of the sequence itself, the Squeeze theorem lets us strategically ignore parts of a function. For instance, since $-1/n \le (\cos n)/n \le 1/n$, we can see that $\lim(\cos n)/n = 0$ without having to deal directly with the sequence $(\cos n)$.

Combining Theorems 9.8 and 9.9, we obtain a very useful computational device:

COROLLARY 9.10: *If* $|x_n - L| \le b_n$ *and* $\lim b_n = 0$, *then* $\lim x_n = L$.

PROOF: Left as Exercise 9.4.5 ■

EXERCISES 9.4

1. Show that each inequality in the hypotheses of Theorem 9.9 need only hold *eventually* for the conclusions to be true.

2. (a) Show that Theorem 9.9.a can't be changed to "If (x_n) converges and $x_n > 0$ for all n, then $\lim x_n > 0$."

   (b) What conclusion can be drawn if it is known that $x_n > 0$?

   (c) Give a condition that would guarantee that $\lim x_n > 0$.

3. Draw a picture to illustrate the proof of Theorem 9.9.b.

4. (a) Show *in two ways* that, if $x_n \ge a$ for all n, and (x_n) converges,

then $\lim x_n \geq a$: (1) Do a proof like that of Theorem 9.9.a; (2) Use the result of Theorem 9.9.b.

(b) Show that the assumption in Theorem 9.9.c that (x_n) and (y_n) have the same limit is necessary. What, if anything can be said in this case if $\lim x_n \neq \lim y_n$?

5. Prove Corollary 9.10.

6. (a) Recall that a rational number p/q is positive if p and q are either both natural numbers or both additive inverses of natural numbers. Show that the following defines a positive set on $\mathbf{R}$:
$\{x \in \mathbf{R} : x \neq 0 \text{ and } \exists (r_n) \ni (r_n \in \mathbf{Q}, r_n > 0, \text{ and } \lim r_n = x)\}$.

(b) Explain why it is acceptable to use the expression $r_n > 0$ in what purports to be a *definition* of a positive set.

(c) Show that it is necessary to include the condition $x \neq 0$ in the definition in (a) [that is, show that the set is *not* a positive set if that condition is left out].

9.5 SEQUENCES AND ALGEBRA

Limits of sequences respect the simple algebraic operations:

THEOREM 9.11: *Suppose* $\lim x_n = L$ *and* $\lim y_n = M$. *Then*

(a) $\lim(x_n + y_n) = L + M$.

(b) $\lim(cx_n) = cL$ *for any constant* c.

(c) $\lim(x_n y_n) = LM$.

(d) $\lim(x_n/y_n) = L/M$ *provided* y_n *is never 0 and* $M \neq 0$.

PROOF: We prove (a) and leave the rest as Exercise 9.5.1. Let $\varepsilon > 0$ be given. There is a number N_1 so that $x_n \in (L-\varepsilon/2, L+\varepsilon/2)$ when $n > N_1$ (The reason for using $\varepsilon/2$ here will be apparent momentarily.) Likewise, there is a number N_2 so that $y_n \in (M-\varepsilon/2, M+\varepsilon/2)$ when $n > N_2$. Let $N = \max\{N_1, N_2\}$. If $n > N$, both these conditions hold, and we have $L+M-\varepsilon = L-\varepsilon/2+M-\varepsilon/2 < x_n+y_n < L+\varepsilon/2+M+\varepsilon/2 = L+M+\varepsilon$, and so $\lim(x_n + y_n) = L + M$. ∎

Theorem 9.11 provides an easy proof of Theorem 9.9.b: If $x_n \leq y_n$ for all n, then $y_n - x_n \geq 0$, and so $\lim(y_n - x_n) \geq 0$. According to Theorem 9.11, $\lim(y_n - x_n) = \lim y_n - \lim x_n$, and the result follows. We can't ignore the hypothesis of Theorem 9.11. We simply can't say that $\lim(x_n + y_n) = \lim x_n + \lim y_n$ unless we know (x_n) and (y_n) both converge.

EXERCISES 9.5

1. (a) Complete the proof of Theorem 9.11.

   (b) Is it necessary to state *both* of the added conditions in part (d) of Theorem 9.11?

2. (a) Give an example to show that it is possible to have $\lim(x_n + y_n)$ exist without having $\lim x_n$ or $\lim y_n$ exist.

   (b) Give an example to show that it is possible to have $\lim(x_n y_n)$ exist without having $\lim x_n$ or $\lim y_n$ exist.

   (c) If $\lim(x_n + y_n)$ exists and $\lim x_n$ exists, must it be the case that $\lim y_n$ exists?

   (c) If $\lim(x_n y_n)$ exists and $\lim x_n$ exists, must it be the case that $\lim y_n$ exists?

9.6 SEQUENCES AND TOPOLOGY

The relationship between sequences and topology is found primarily in three topics: cluster points, closed sets, and continuous functions. We begin with a lemma that makes a simple but important observation. We say a point is a **cluster point** of a sequence if it is a cluster point of its range.

LEMMA 9.12: *A convergent sequence can have at most one cluster point, its limit.*

PROOF: Suppose (x_n) converges to L and that $a \neq L$. We will show that a is not a cluster point of $\{x_n\}$. Let $\varepsilon = |a - L|/2$. Then the intervals $(L - \varepsilon, L + \varepsilon)$ and $(a - \varepsilon, a + \varepsilon)$ are disjoint. Now there can be only finitely many values of n for which $x_n \notin (L - \varepsilon, L + \varepsilon)$. This means that $\{x_n\} \cap (a - \varepsilon, a + \varepsilon)$ is finite, and a is not a cluster point of (x_n). ∎

Lemma 9.12 *doesn't* say that a cluster point of a sequence is necessarily its limit, or even that the limit of a convergent sequence is necessarily a cluster point. The latter is, however, one of only two possibilities:

THEOREM 9.13: *If* $\lim x_n = L$, *one of the following is true:*

(i) (x_n) *is eventually equal to* L

or (ii) L *is the only cluster point of* (x_n).

PROOF: This statement is of the form $A \Rightarrow (B \text{ or } C)$. This is equivalent to $(A \text{ and not } B) \Rightarrow C$. Suppose that $\lim x_n = L$ and that (x_n) is not

eventually equal to L. Let $\varepsilon > 0$ be given. Since (x_n) is not eventually equal to L, there are infinitely many values of n for which $x_n \neq L$. But since (x_n) converges to L, it is outside the interval $(L-\varepsilon, L+\varepsilon)$ for only finitely many values of n. Consequently, there must be a value of n for which x_n is different from L but x_n is in the interval $(L-\varepsilon, L+\varepsilon)$. Thus L is a cluster point of (x_n). By Lemma 9.12, L is the only cluster point of (x_n). ∎

The converse of Theorem 9.13 is only partially true. It is easy to see that (x_n) converges to L if it is eventually equal to L (and notice that such a sequence has no cluster points). On the other hand, a sequence might have only one cluster point yet still not converge: Let $x_n = 1$ if n is odd and $1/n$ if n is even. Then 0 is the only cluster point of (x_n), but (x_n) doesn't converge. We will see shortly that our inability to obtain a definitive statement along these lines is a result not so much of the intractability of the problem, but of our not looking at it in quite the right way.

LEMMA 9.14: *The point c is a cluster point of a set S if and only if there is a sequence of elements of S, all different from c and converging to c.*

PROOF: The "if" part follows from Exercise 7.5.2 and Theorem 9.13. Now suppose c is a cluster point of S. For each n there is an element of S, say x_n, so that $x_n \in (c - 1/n, c + 1/n)\backslash\{c\}$. By Corollary 9.10, the sequence (x_n) does what we want. ∎

This proof of Lemma 9.14 uses the Archimedean property. On a deeper level, the result relies on the fact that the topology of the real number line is first countable (see Exercise 8.4.6). There are topological spaces in which neither the Archimedean property nor first countability hold, and Lemma 9.14 also is not true in every topological space. The next two theorems show us the real substance of the relationship between sequences and topology.

THEOREM 9.15: *A subset S of* **R** *is closed if and only if $\lim x_n \in S$ whenever (x_n) is a convergent sequence whose terms are all in S.*

PROOF: We will do half of the proof by the forward-backward method. Here there is more "forward" than "backward." Remember that this is a universally quantified statement, which explains the quick insertion of the second step below.

▽▽

Let S be closed.

$\hookrightarrow$ Let (x_n) be a convergent sequence whose terms are in S.

⋆ ⋆ ⋆

$\lim x_n \in S$.

△△

We have limited information about convergent sequences, and so we can quickly review it all. In view of the definition of "closed," Theorem 9.13 looks useful:

▽▽

Let S be closed.

Let (x_n) be a convergent sequence whose terms are in S.

$\hookrightarrow$ $\lim x_n$ either is a cluster point of $\{x_n\}$ or is an element of $\{x_n\}$.

⋆ ⋆ ⋆

$\lim x_n \in S$.

△△

Our working *hypothesis* is now a statement of the form A or B, and so we should do a proof by cases. Remember: The cases *must exhaust all possibilities*, and each case *must lead to the desired conclusion.* Theorem 9.13 guarantees that we have exhausted all possibilities in the following:

▽▽

Let S be closed.

Let (x_n) be a convergent sequence whose terms are in S.

$\lim x_n$ either is a cluster point of $\{x_n\}$ or is an element of $\{x_n\}$.

$\hookrightarrow$ **CASE 1:**	$\hookrightarrow$ **CASE 2:**
$\hookrightarrow$ $\lim x_n$ is a cluster point of $\{x_n\}$.	$\hookrightarrow$ $\lim x_n \in \{x_n\}$.
⋆ ⋆ ⋆	⋆ ⋆ ⋆
$\lim x_n \in S$.	$\lim x_n \in S$.

△△

We observe that $\{x_n\} \subseteq S$ and use Exercise 7.5.2 for the first case. The second case is even easier:

▽▽

Let S be closed.

Let (x_n) be a convergent sequence in S.

$\lim x_n$ either is a cluster point of $\{x_n\}$ or is an element of $\{x_n\}$.

Case 1:	**CASE 2:**
$\lim x_n$ is a cluster point of $\{x_n\}$.	$\lim x_n \in \{x_n\}$.

$\hookrightarrow \{x_n\} \subseteq S$. $\hookrightarrow \lim x_n$ is a cluster point of S. $\lim x_n \in S$.	$\hookrightarrow \{x_n\} \subseteq S$. $\lim x_n \in S$.

Now suppose c is a cluster point of S. By Lemma 9.14, there is a sequence of elements of S converging to c. By hypothesis, the limit of any convergent sequence in S is in S, and so $c \in S$ and S is closed. ∎

We could define closed sets by specifying which sequences converge, then define open sets using Theorem 8.10. In this sense, the collection of convergent sequences on the real number line carries the same information as its topology (though this is not the case for all topological spaces).

THEOREM 9.16: *A function $f : \mathbf{R} \to \mathbf{R}$ is continuous if and only if, for each convergent sequence (x_n), $\lim f(x_n) = f(\lim x_n)$.*

PROOF: Suppose f is continuous and let $L = \lim x_n$ and $\varepsilon > 0$ be given. The interval $I = (f(L) - \varepsilon, f(L) + \varepsilon)$ is an open set containing $f(L)$. Thus $U = f^{-1}(I)$ is an open set containing L, and so U is a neighborhood of L. Since (x_n) converges to L, it is eventually in U, and so $(f(x_n))$ is eventually in I, that is, $(f(x_n))$ converges to $f(L)$. In the other direction, we use an ε-δ argument since the definition of "not continuous" is more easily stated in those terms. If f is not continuous, there is an $L \in \mathbf{R}$ so that f is not continuous(ε) at L. Then it is *not* the case that

$$\forall \varepsilon > 0 \exists \delta > 0 \ni \forall x (|x - L| < \delta \Rightarrow |f(x) - f(L)| < \varepsilon),$$

in other words,

$$\exists \varepsilon > 0 \ni \forall \delta > 0 \exists x \ni (|x - L| < \delta \text{ and } |f(x) - f(L)| \geq \varepsilon).$$

Let $\varepsilon > 0$ be provided by this statement, and for each natural number n, let x_n be a number with $|x_n - L| < 1/n$ and $|f(x_n) - f(L)| \geq \varepsilon$. Then (x_n) converges to L, but $(f(x_n))$ doesn't converge to $f(L)$. ∎

EXERCISES 9.6

1. Draw a picture to illustrate the proof of Lemma 9.12.

2. (a) Complete the proof of Lemma 9.14 and explain how the Archimedean property is used.

   (b) Construct a proof of Lemma 9.14 based on the first countability of the topology of the real line.

3. Use Theorem 9.15 to show that the union and intersection of two closed sets are closed.

4. Suppose S is a nonempty open set that isn't the whole real line. Show that there is a sequence of elements of S that converges to an element of $C(S)$.

5. Adjust Theorem 9.16 to account for functions whose domains are not the whole real line. Prove your new theorem.

6. (a) Show that a sequence that is eventually constant converges.

   (b) Show that if the topological space X has the discrete topology, the *only* sequences that converge are those that are eventually constant.

   (c) Show that if the topological space X has the indiscrete topology, *every* sequence converges to *every* element of X (!)

9.7 SUBSEQUENCES

Suppose we've found that a sequence diverges. Can we say anything more about it? The sequence $((-1)^n)$ diverges, but it has a "subsequence" (we will give a precise definition of this in a moment) that goes 1, 1, 1, ..., and certainly converges. The sequence (n) also diverges, but any sequence we might construct by selecting terms from (n) will also diverge. Both $((-1)^n)$ and (n) diverge, but their subsequences behave differently.

DEFINITION 9.17: (a) A function $n : \mathbf{N} \to \mathbf{N}$ is said to be **strictly increasing** if $n(k+1) > n(k)$ for all $k \in \mathbf{N}$.

(b) If $x : \mathbf{N} \to \mathbf{R}$ is a sequence and $n : \mathbf{N} \to \mathbf{N}$ is strictly increasing, then $x \circ n : \mathbf{N} \to \mathbf{R}$ is called a **subsequence** of x.

The notation of Definition 9.17 can be confusing, and we won't use it much. Instead of writing $n(k)$ for the values of the function n, we will write n_k (note that the function n is, in effect, a sequence of natural numbers). The subsequence $(x(n(k)))$ is then written (x_{n_k}). Observe that the index of this subsequence is k, not n_k or n. Since $n_{k+1} > n_k$ for all k, a subsequence of (x_n) consists of infinitely many terms of (x_n) *kept in the same order*. It will be useful to note that, if n is as in the definition, then $n_k \geq k$ for all k.

EXAMPLES 9.7: 1. In the example of $x_n = (-1)^n$, we may take $n_k = 2k$ to get the subsequence $1, 1, \ldots$ and $n_k = 2k+1$ to get $-1, -1, \ldots$. Note that a divergent sequence can have convergent subsequences.

2. Let $x_n = 1/n$ and $n_k = k^2$. Then $x_{n_k} = 1/k^2$. It appears that this converges to 0, as does the original sequence.

Is there a relationship between convergence of a sequence and convergence of its subsequences? We saw in Example 1 that convergence of some subsequences doesn't guarantee the convergence of the sequence, but...

THEOREM 9.18: *If the sequence (x_n) converges, so does every subsequence of it, and all converge to the same limit.*

PROOF: Suppose that (x_n) converges to L and (x_{n_k}) is a subsequence. Let V be a neighborhood of L. Let N be such that $x_n \in V$ whenever $n > N$. If $k > N$, we have $n_k \geq k > N$, and so $x_{n_k} \in V$, and (x_{n_k}) converges to L. ∎

COROLLARY 9.19: *If a sequence has subsequences that converge to two different limits, the sequence diverges.* ∎

This provides a much quicker proof that $((-1)^n)$ diverges (it has subsequences converging to both 1 and -1). We will see in the next chapter that even partial converses to Theorem 9.18 are difficult to come by.

EXERCISES 9.7

1. Show that if Y is a subsequence of X and Z is a subsequence of Y, then Z is a subsequence of X.

2. (a) If $g : \mathbf{N} \to \mathbf{N}$ is strictly increasing, show that $g(k) \geq k$.

   (b) A function $g : \mathbf{R} \to \mathbf{R}$ is strictly increasing if $x > y \Rightarrow g(x) > g(y)$. Show that even if $g : \mathbf{R} \to \mathbf{R}$ is strictly increasing, it is not necessarily the case that $g(x) \geq x$ for all x.

3. Discuss the behavior of the sequence $(\cos n)$ and its subsequences.

4. Use Theorem 9.16 and Corollary 9.19 to show that $f(x) = \sin(1/x)$ can't be defined at 0 in such a way to make it continuous.

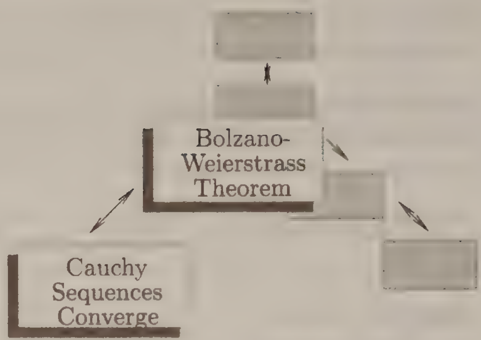
5. (a) Show that it is possible for a sequence to have no convergent subsequences at all.

   (b) Show that it is possible to have a sequence (x_n) diverge while the sequence $(|x_n|)$ converges.

   (c) Is it possible to have a sequence (x_n) with no convergent subsequences, yet have $(|x_n|)$ converge?

Chapter 10

Sequences and the Big Theorem



10.1 CONVERGENCE WITHOUT LIMITS

We know that $\lim 1/n = 0$ and that the sequence we found at the beginning of the previous chapter converges. Our knowledge of the latter sequence, however, is quite different from our knowledge of the former. Though we can estimate the difference between the terms of the latter sequence and its limit, we don't know exactly what that limit is (though we certainly have strong suspicions). Our definitions of convergence are designed to allow us to check whether a guess really is the limit of a sequence, but give us no hint how to find those guesses in the first place. In this chapter we develop methods by which we can sometimes decide whether a sequence converges without ever knowing its limit. It is quite remarkable that we can even hope to do this. These results are among the deepest we see in this book, and have the most important ramifications in applied mathematics.

As we would expect, the first of these theorems applies only to the simplest sorts of sequences. Here we will also see the deeper aspects of the relationship between convergence of sequences and the order structure of the real numbers.

10.2 MONOTONE SEQUENCES

DEFINITION 10.1: A sequence (x_n) is **increasing** if $x_{n+1} \geq x_n$ for each n, and **decreasing** if $x_{n+1} \leq x_n$ for each n. A sequence is **monotone** if it is either increasing or decreasing.

Note the inequality symbols carefully. The sequence given by $1, 1, \ldots$, is *both* increasing and decreasing. If it happens that $x_{n+1} > x_n$ for each n, we say (x_n) is **strictly increasing**, and similarly for the other terms.

EXAMPLES 10.2: 1. The sequence $(1/n)$ is strictly decreasing since $1/(n+1) < 1/n$ for all n. The sequence $(1 - 1/n)$ is strictly increasing.

2. The sequence $0, 1, 1/2, 1/3, \ldots$ is not monotone, even though it's not very different from $(1/n)$. It is not increasing because $1/2 = x_3 < x_2 = 1$ and is not decreasing because $1 = x_2 > x_1 = 0$. It is, however, *eventually* decreasing.

3. The sequence $((-1)^n)$ is not monotone and not eventually monotone. If n is odd, $-1 = (-1)^n < (-1)^{n+1} = 1$ and $1 = (-1)^{n+1} > (-1)^{n+2} = -1$. Since we can find odd numbers as large as we want, the sequence is not eventually monotone.

4. The sequence constructed at the beginning of Chapter 9 to estimate $\sqrt{2}$ is increasing (but not strictly increasing). The process described there can never result in a guess that is to the left of the previous one.

The following is the first step toward our goal of deciding whether a sequence converges without having to know its limit. It is sometimes called the **Monotone Convergence Theorem**, but BEWARE, there is another theorem by this name down the road in your mathematical career (it is about a different subject and much deeper). We pick out a section of the proof as a lemma. It was used in Theorem 7.6, but its proof was put off because we now have better terminology. You should check to see that there is no circular reasoning here, that is, that there is no use made of Theorem 7.6 in this proof. This is always a risk when the proof of a result is so far removed from the place it is used.

LEMMA 10.2: (a) *Any cluster point of an increasing sequence is an upper bound for the sequence.*

(b) *Any cluster point of a decreasing sequence is a lower bound for the sequence.*

PROOF: We will prove (a). Call the sequence (x_n) and suppose that $x < x_{n_0}$ for some n_0. Notice that $(-\infty, x_{n_0})$ is a neighborhood of x but $\{x_n\} \cap (-\infty, x_{n_0})$ is finite (it has at most $n_0 - 1$ elements). Thus x is not a cluster point of $\{x_n\}$. Note in particular that no element of $\{x_n\}$ is a cluster point of $\{x_n\}$. ∎

THEOREM 10.3: *A bounded, monotone sequence converges.*

PROOF: Let (x_n) be a bounded, monotone sequence. We may suppose that (x_n) is increasing. Its range is either finite or infinite. If its range is

finite, (x_n) is eventually constant (you will verify this in Exercise 10.2.6), and so it converges. If $\{x_n\}$ is infinite, it has a cluster point by the Bolzano-Weierstrass theorem. Let u be a cluster point of $\{x_n\}$ and let $\varepsilon > 0$ be given. By Lemma 10.2, $u > x_n$ for all n. By the definition of cluster point, there is an element of $\{x_n\}$, say x_N, in the interval $(u - \varepsilon, u + \varepsilon)$. Then for all $n > N$, we have $x_N \leq x_n < u$. Then $x_n \in (u - \varepsilon, u + \varepsilon)$ for all such n and (x_n) converges to u. ∎

By Theorem 5.3, $u = \sup\{x_n\}$, allowing us to sharpen this result a bit:

COROLLARY 10.4: *If (x_n) is increasing and bounded, then $\lim x_n = \sup\{x_n\}$. If (x_n) is decreasing and bounded, then $\lim x_n = \inf\{x_n\}$.*

PROOF: Left as Exercise 10.2.5. This is a corollary as much to the proof of Theorem 10.3 and to Theorem 5.3 as to Theorem 10.3 itself. ∎

EXERCISES 10.2

1. Which of these is the definition of "monotone"? What does the other say about the sequence (x_n)?

   (i) $\forall n(x_n \geq x_{n+1} \text{ or } x_n \leq x_{n+1})$.

   (ii) $\forall n(x_n \geq x_{n+1}) \text{ or } \forall n(x_n \leq x_{n+1})$.

2. Complete the proof of Lemma 10.2 and verify the comment at the end of its partial proof.

3. Let $s_n = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2}$. Show that $s_n < 2 - \frac{1}{n}$ for $n = 2, 3, \ldots$. What does this say about $\lim s_n$?

4. Show that the sequence $(\frac{100^n}{n!})$ is eventually monotone. Is it eventually increasing or decreasing? Find the value of N after which it is monotone. Does it converge?

5. Prove Corollary 10.4.

6. (a) Show that a sequence that is eventually constant has a finite range.

   (b) Show that the converse of this is not true (a sequence with a finite range is not necessarily eventually constant).

   (c) Show that a *monotone* sequence with a finite range is eventually constant.

7. If S is a bounded set, show that there is an increasing sequence of elements of S converging to $\sup S$ and a decreasing sequence of elements of S converging to $\inf S$

8. Suppose f is an increasing, bounded function whose domain contains some ray $[a, \infty)$. Show that $\lim_{x\to\infty} f(x)$ exists.

9. Define H and G by $h_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ and $g_n = 1 + \frac{1}{2} + \cdots + \frac{1}{2^n}$.

   (a) Show that H and G are both increasing.

   (b) Show that H diverges and G converges.

   (c) What is $\lim g_n$?

   (d) Suppose (a_n) is a sequence of positive numbers and let $S = (s_n)$ be defined by $s_n = a_1 + a_2 + \cdots + a_n$. Show that S converges if and only if it is bounded.

   (e) Suppose (a_n) and (b_n) are sequences of positive numbers, with $a_n \leq b_n$ for all n. Let S and T be constructed from (a_n) and (b_n), respectively, as in (d). Show:

   (i) If T converges then S converges
   (ii) If S diverges then T diverges
   and (iii) It is possible for S to converge and T to diverge.

   (f) (Adventure!) Show that $\lim(h_n - \ln(n))$ exists. [Hints: Think about integrals, Riemann sums, and rectangles. You will need to know that the sequence defined as in (d) with $a_n = 1/n^2$ converges.] This limit is called **Euler's Constant**. It is denoted γ, and its value is approximately 0.577. No one knows whether γ is rational or irrational!

10.3 A RECURSIVELY DEFINED SEQUENCE

Define a sequence like this: Let $x_1 = 1$ and $x_{n+1} = \sqrt{3 + x_n}$ for $n \geq 1$. This is called a **recursively defined** sequence. Recursive procedures are useful in a variety of settings. They can often be used to approximate a solution where an exact one is difficult or impossible to find. They also lend themselves nicely to proofs by induction. We will show that (x_n) converges. Its first few terms are $1, 2, \sqrt{5} \cong 2.236, \sqrt{3 + \sqrt{5}} \cong 2.288, \ldots$. The sequence seems to be increasing. We can prove this by induction:

$P(n) : x_{n+1} > x_n$
$P(1) : x_2 = 2 > 1 = x_1$

Assume $P(k) : x_{k+1} > x_k$
Want $P(k+1) : x_{k+2} > x_{k+1}$

$$\begin{aligned} x_{k+2} &= \sqrt{3 + x_{k+1}} \\ &> \sqrt{3 + x_k} \qquad \text{(since } \sqrt{x} \text{ is an increasing function, and using the induction hypothesis)} \\ &= x_{k+1}. \end{aligned}$$

A little more work with the calculator suggests that the sequence grows very slowly. For instance, x_{10} is only about 2.3. (A calculator is often a good place to look for ideas about sequences, but one must be sure to prove that those guesses are correct.) To be cautious, we might guess that all the terms are less than 3. This, too, can be proved by induction.

$P(n): x_n < 3$
$P(1): x_1 = 1 < 3$

Assume $P(k): x_k < 3$
Want $P(k+1): x_{k+1} < 3$

$$\begin{aligned} x_{k+1} &= \sqrt{3+x_k} \\ &< \sqrt{3+3} && \text{(induction hypothesis; } \sqrt{x} \text{ is increasing)} \\ &= \sqrt{6} \\ &< 3. \end{aligned}$$

Thus (x_n) is bounded and increasing, and so it converges. What is its limit? Knowing that a sequence has a limit can be a big step toward finding it. Call the limit L. If we make n very large, x_n is very close to L (we gloss over some details here). Since $1 < L \leq 3$, $\sqrt{x}$ is continuous at $3 + L$, and so if x_n is very close to L, $x_{n+1} = \sqrt{3 + x_n}$ is very close to $\sqrt{3+L}$. But x_{n+1} is also very close to L, and so $L \cong x_{n+1} = \sqrt{3+x_n} \cong \sqrt{3+L}$ (and $\cong$ can be made as close to $=$ as we like). The limit of this sequence is between 1 and 3 and is a solution to $L = \sqrt{3+L}$. This means $L = (1+\sqrt{13})/2 \cong 2.303$.

Sequences like the one in Example 10.2.2 can make application of the Monotone Convergence theorem difficult. It seems, though, that the behavior of a sequence near the beginning should not affect its convergence or divergence. Adjusting the proof of Theorem 10.3, we may show:

THEOREM 10.5: *A sequence that is bounded and eventually monotone converges.* ∎

This is not the whole story since not every convergent sequence is eventually monotone. The sequence $((-1)^n/n)$ converges to 0 (Exercise 9.3.5) but is not eventually monotone.

EXERCISES 10.3

1. Prove Theorem 10.5.

2. Say how we know, in the example in this section, that $1 < L \leq 3$.

3. Suppose that $F : \mathbf{R} \to \mathbf{R}$, that $x_1 = y_1$, and that $x_{n+1} = F(x_n)$ and $y_{n+1} = F(y_n)$. Show that $x_n = y_n$ for all n.

4. Prove that the behavior of a sequence near the beginning doesn't affect its convergence or divergence in the following way. For a sequence $X = (x_n)$, let $X(m)$ be the sequence given by $x(m)_n = x_{m+n}$. Show that X converges if and only if $X(m)$ converges for every $m \in \mathbf{N}$. (The hardest part of this is understanding the notation!)

5. Let $x_1 = a > 0$ and $x_{n+1} = x_n + \frac{1}{x_n}$. Show that (x_n) diverges.

6. Suppose $a > 0$ and $z_1 > 0$. Let a sequence be defined recursively by $z_{n+1} = (a + z_n)^{1/2}$ for $n \geq 1$. Discuss the convergence of this sequence for various values of a.

7. Let the sequence (s_n) be given by $s_1 = \sqrt{2}$ and $s_{n+1} = \sqrt{2 + \sqrt{s_n}}$ for $n = 1, 2, \ldots$. Show that (s_n) converges and find its limit.

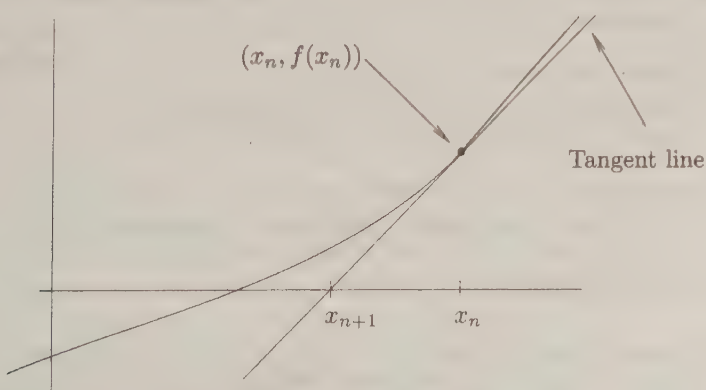
8. **Newton's Method** for finding square roots is described by

$$x_1 = 1; \quad x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \text{ for } n \geq 1$$

(a) If $a > 0$, show that (x_n) is a bounded, eventually monotone sequence (the only term that might not be "in order" is the first one).

(b) Show that $\lim x_n = \sqrt{a}$.

9. The general Newton's method is a recursive procedure for approximating the solution to an equation $f(x) = 0$. The means of obtaining a guess x_{n+1} from the previous guess x_n is described in the following picture:



The point x_{n+1} is the intersection of the tangent line to $y = f(x)$ at

the point $(x_n, f(x_n))$ with the x-axis.

(a) Beginning with this diagram, show that $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

(b) Note that $\sqrt{a}$ is a solution to $x^2 - a = 0$. Verify the formula in Exercise 10.3.8.

(c) Find Newton's method for cube roots.

(d) If (i) $f(x)$ is strictly monotone and concave up on an interval that contains both the solution and the initial guess x_1, (ii) $f' > 0$ at the solution, and (iii) the initial guess is greater than the solution (as in the picture), show that (x_n) is decreasing and bounded.

(e) If the conditions in (d) hold, show that $\lim x_n$ is the solution to the equation $f(x) = 0$.

(f) The formula for Newton's method suggests problems if $f'(x_n) = 0$ for any n. Why? Draw some sketches to see what might happen if f' is close to or equal to 0 anywhere between the initial guess and the solution.

(g) When Newton's method works, the convergence of the guesses to the solution can be quite rapid. Using $x_1 = 2$ as the initial guess, apply Newton's method for square roots 5 or 6 times, and observe the way the *error* changes (you'll be doing this on a calculator, which can also give you the value of $\sqrt{2}$). Is there a pattern in the way the errors decrease? Subtract each error from the previous one. Divide each error by the previous one. Count decimal places of accuracy. Make a conjecture. Does the same pattern hold for the cube root method you found in (c)?

(h) Apply Newton's method (with a calculator) to a problem for which you *don't* know the answer. (For instance, use it to find the solution to $\cos(x) = x$.) Can you relate the pattern of the guesses to what you discovered in (g)?

10. Define the sequence (x_n) by $x_1 = 1$ and $x_{n+1} = \frac{3}{x_n^2}$ for $n \geq 1$. The procedure described in the section suggests that $\lim x_n = \sqrt[3]{3}$. Is this true?

11. The point a is called a **fixed point** of the function f if $f(a) = a$. If a convergent sequence is defined by $x_1 = a$ and $x_{n+1} = f(x_n)$, and if f is continuous at the limit of the sequence, show that the limit is a fixed point of f.

12. Describe the expression $x^{x^{x^{x}}}$ in terms of a recursive sequence—recall that $a^{b^c} = a^{(b^c)}$. Are there any values of x for which this sequence has a limit? What is that limit?

13. a) Consider the sequence given by $x_1 = 0, x_{n+1} = \frac{1}{2+x_n}$. Show that this sequence converges (consider the subsequences consisting of every other term). What is the limit of this sequence?

(b) If we begin to write these terms out, they look like

$$\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{\cdots}}}}$$

Such a construction is called a **continued fraction**. We may represent a continued fraction by specifying the numbers along the lower left diagonal. For instance, the one above would be represented as $[2, 2, 2, \ldots]$, while $[1, 2, 3, \ldots]$ would give us

$$\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{\cdots}}}}$$

Even though the limit of the sequence in (a) is irrational (and so has a nonrepeating decimal expansion), it does have a repeating continued fraction representation. Find $[1, 1, 1, \ldots]$. The study of continued fractions has given rise to a number of advances in mathematics. Stieltjes was studying continued fractions when he conceived of the Riemann-Stieltjes integral (see Chapter 17), though the connection is tough to pick out.

(c) (Research) Suppose we take a sequence (a_n) and construct from it a continued fraction $[a_1, a_2, \ldots]$. Is there a relationship between the convergence of the sequence and the convergence of the continued fraction?

10.4 THE BOLZANO-WEIERSTRASS THEOREM (REVISITED)

We modify Lemma 9.14 to obtain the following:

THEOREM 10.6: *If c is a cluster point of $\{x_n\}$, there is a subsequence (x_{n_k}) converging to c.*

PROOF: This seems very similar to Lemma 9.14, but if we try the same proof, the elements we find don't necessarily form a subsequence

of (x_n) (they may not be in the proper order). We modify that proof, being careful to keep the elements we select in the proper order. Since $\{x_n\} \cap (c - 1/k, c + 1/k)$ is infinite for each k, so is the set

$$S_k = \{n : x_n \in (c - 1/k, c + 1/k)\}.$$

Let $x_{n_1} \in (c-1, c+1)$. Since S_2 is infinite, there is an $n_2 > n_1$ with $x_{n_2} \in (c - 1/2, c + 1/2)$. Similarly, since S_3 is infinite, there is an $n_3 > n_2$ with $x_{n_3} \in (c - 1/3, c + 1/3)$. Continuing in this way, we find $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$, with $n_{k+1} > n_k$ and $x_{n_k} \in (c - 1/k, c + 1/k)$ for each k. It follows that $\lim x_{n_k} = c$. ∎

In this proof it is more important that the sets S_k are infinite than it is that c is a cluster point of $\{x_n\}$. This suggests a change in our point of view:

DEFINITION 10.7: A point c is a **sequential cluster point** of (x_n) if $\{n : x_n \in (c - \varepsilon, c + \varepsilon)\}$ is infinite for each $\varepsilon > 0$.

Note well that c is a sequential cluster point of (x_n) if the set of *subscripts* for which $x_n \in (c - \varepsilon, c + \varepsilon)$ is infinite for every ε. Every cluster point of a sequence is a sequential cluster point, but the converse is not true. (This is another way mathematics gets done. If we observe that a proof passes through an important intermediate stage, we make a definition to encompass anything satisfying the condition of that stage.)

THEOREM 10.8: *c is a sequential cluster point of (x_n) if and only if (x_n) has a subsequence converging to c.*

PROOF: Left as Exercise 10.4.2. (We may consider this an alternative definition of "sequential cluster point.") ∎

The following, sometimes called the **Bolzano-Weierstrass Theorem for Sequences**, reassures us that we've found an appropriate analogue for cluster points.

THEOREM 10.9: *Every bounded sequence has a sequential cluster point.*

PROOF: Let (x_n) be a bounded sequence. Then $\{x_n\}$ is either finite or infinite. If it is finite, at least one element of it must be repeated for infinitely many values of n. This repeated element, with its subscripts kept in order, is a convergent subsequence. If the range is infinite, the Bolzano-Weierstrass theorem (for sets) guarantees that the range has an

ordinary cluster point. Such a point is also a sequential cluster point by Theorems 10.6 and 10.8. ∎

Combining the last two theorems, we obtain the following, also sometimes called the Bolzano-Weierstrass theorem for sequences.

COROLLARY 10.10: *Every bounded sequence has a convergent subsequence.* ∎

EXERCISES 10.4

1. Prove the comment preceding Theorem 10.8.

2. Prove Theorem 10.8.

3. Give an example of a sequence (x_n) having both 1 and 2 as sequential cluster points but such that $\{x_n\}$ has only 1 as a cluster point.

4. Find all sequential cluster points of the sequence $(\sin(n))$. (Very difficult!)

5. Show that a bounded sequence diverges if and only if it has (at least) two sequential cluster points.

6. (a) Show that every sequence has a monotone subsequence.

   (b) Use this to prove the Bolzano-Weierstrass theorem for sequences.

7. Prove directly that a monotone sequence can have only one sequential cluster point. (In this case, "directly" means that you shouldn't refer to Theorem 10.5.)

8. Verify the following statement from the proof of Theorem 10.9: "If [the range] is finite, at least one element of it must be repeated for infinitely many values of n."

9. Show that a sequence with an ordinary cluster point can't be eventually constant.

10. (a) The **limit superior** of a sequence (a_n), denoted $\limsup a_n$, is the supremum of its set of *sequential* cluster points [if (a_n) is not bounded above, $\limsup a_n = \infty$]. Show that $\limsup a_n = \lim_{n\to\infty} \sup_{k\geq n}\{a_k\}$ (hence the name).

    (b) The limit superior of a *set* was defined in Exercise 7.6.6. Show that $\limsup a_n$ is not necessarily the same as $\limsup\{a_n : n \in \mathbf{N}\}$.

(c) State a definition and repeat (a) for the **limit inferior** of a sequence.

(d) Show that $\lim a_n = L$ if and only if $\liminf a_n = \limsup a_n = L$.

(e) Show that in general, $\limsup(a_n + b_n) \neq \limsup a_n + \limsup b_n$. Is there any predictable relationship between the sides of this expression?

(f) Show that a bounded sequence (a_n) has a subsequence that converges to $\limsup a_n$. (This provides another proof of the Bolzano-Weierstrass theorem for sequences, because every bounded sequence has a lim sup.)

(g) Suppose (a_n) is a bounded sequence with $a_n \geq a$ for all n and $\limsup a_n = a$. Show that $\lim a_n = a$.

(h) Show that it is possible to have $\limsup a_n = \inf a_n$ and it is possible to have $\liminf a_n = \sup a_n$. Is it possible for both of these to happen for the same sequence?

11. (a) Let $r_1, r_2, \ldots,$ be an enumeration of the rational numbers in the interval $[0, 1]$. Show that every element of $[0, 1]$ is a sequential cluster point of (r_n).

    (b) Is there a sequence whose set of sequential cluster points is $(0, 1)$?

12. Show that if a sequence has a bounded subsequence, it also has a convergent subsequence.

13. If X and Y are sequences, define their "weave" $X \varpi Y$ to be the sequence given by $x_1, y_1, x_2, y_2, \ldots,$ and denote the set of sequential cluster points of X by X'. Show that $(X \varpi Y)' = X' \cup Y'$.

10.5 THE CONVERSE OF THEOREM 9.18?

The converse of Theorem 9.18 would say "If every subsequence of (x_n) converges to L, so does (x_n)." This is such a weak statement that it says nothing of interest. Useful partial converses of Theorem 9.18 are hard to come by. The Bolzano-Weierstrass theorem for sequences gives us the following. Sometimes a slight weakening of a useless statement can make it into a useful one.

THEOREM 10.11: *If (x_n) is a bounded sequence with the property that every convergent subsequence converges to the same number L, then (x_n) converges to L.*

PROOF: Since (x_n) is bounded, there is an interval $[-B, B]$ that contains $\{x_n\}$. Suppose that (x_n) doesn't converge to L. Then there is an

$\varepsilon > 0$ so that (x_n) is *not* eventually in $V = (L-\varepsilon, L+\varepsilon)$. Then there is a subsequence (x_{n_k}) with $x_{n_k} \in [-B, B]\backslash V$ for all k. Now (x_{n_k}) is bounded and so has a sequential cluster point, say c. Since $\{x_{n_k}\} \subseteq [-B, B]\backslash V$ and $[-B, B]\backslash V$ is closed, it must be that $c \in [-B, B]\backslash V$. In particular, $c \neq L$. Any sequential cluster point of (x_{n_k}) is also a sequential cluster point of (x_n), and so $c = L$, a contradiction. ∎

EXERCISES 10.5

1. Why is Theorem 10.11 a *weakening* of the converse of Theorem 9.18?

2. Does Theorem 10.11 hold if the sequence isn't assumed to be bounded?

3. Suppose (x_n) is a bounded sequence such that every subsequence of (x_n) has a subsequence that converges to L. Show that (x_n) converges to L.

10.6 CAUCHY SEQUENCES

While examining the sequence $((-1)^n)$, we noticed that *if the terms in a sequence get close to a limit, they must get close to each other.* In one of life's ironic twists, the deepest of ideas springs from this simple observation.

DEFINITION 10.12: The sequence (x_n) is a **Cauchy sequence** if, for any $\varepsilon > 0$, there is a natural number N so that $|x_m - x_n| < \varepsilon$ whenever $m > N$ and $n > N$.

The definition of a Cauchy sequence is a precise statement of the phrase "the terms in the sequence get close to each other." Unfortunately, this concept does not lend itself easily to topological interpretation.

Because $|a - b| = |b - a|$, it doesn't matter which of m or n is larger, and so we may say "$|x_m - x_n| < \varepsilon$ whenever $m \geq n > N$," if it is convenient.

EXAMPLES 10.6: 1. $(\frac{1}{n})$ is a Cauchy sequence. Let $\varepsilon > 0$ be given, and let N be such that $1/N < \varepsilon$. If $m \geq n > N$, then

$$|x_m - x_n| = \frac{1}{n} - \frac{1}{m} < \frac{1}{n} < \frac{1}{N} < \varepsilon.$$

2. Let $x_n = 1 + \dfrac{1}{2} + \dfrac{1}{4} + \cdots + \dfrac{1}{2^n}$ and let $m \geq n > N$. Then

$$
\begin{aligned}
&|x_m - x_n| \\
&= \frac{1}{2^m} + \cdots + \frac{1}{2^{n+1}} \\
&= \frac{1}{2^{n+1}}\left(1 + \cdots + \frac{1}{2^{m-n-1}}\right) \\
&= \frac{1}{2^{n+1}}\left(2 - \frac{1}{2^{m-n-1}}\right) \\
&< \frac{1}{2^n} \\
&< \frac{1}{2^N},
\end{aligned}
$$

which can be made as small as we like (you should check all of these statements).

3. $(\ln(n))$ is *not* a Cauchy sequence. Let $m \geq n$. Examine $\ln(m) - \ln(n)$. By the Mean Value theorem,[1] this is equal to $(m - n)(1/c)$ for some c between m and n. Then $1/c > 1/m$, and so $(m - n)(1/c) > (m - n)/m = 1 - n/m$. This approaches 1 for any n. Thus $\ln(m) - \ln(n)$ can't be made close to 0 in the sense of Cauchy sequences. Notice, however, that $\ln(n) - \ln(n - 1)$ *does* go to 0 as $n \to \infty$. The latter condition $(x_n - x_{n-1} \to 0)$ is *not* sufficient to guarantee that a sequence is a Cauchy sequence.

4. The sequence constructed in the example at the beginning of Chapter 9 is a Cauchy sequence. If we let I_n be the current interval at the nth stage of that process, observe that $x_k \in I_n$ whenever $k > n$. By Theorem 4.19, $|x_k - x_j| \leq$ (the length of I_n) $= 1/2^n$ if $j, k > n$. Since $\lim 1/2^n = 0$, (x_n) is a Cauchy sequence. This example shows that approximation procedures can sometimes be shown to produce Cauchy sequences even when the exact solution to a problem is not known.

Earlier, we observed that "if the terms in a sequence get close to a limit, they must get close to each other," and this led us to the idea of Cauchy sequences. We may now prove this observation.

THEOREM 10.13: *A convergent sequence is a Cauchy sequence.*

PROOF: Let (x_n) converge to L and let $\varepsilon > 0$ be given. Let N be such that $|x_n - L| < \varepsilon/2$ whenever $n > N$. If $m, n > N$, then

---

[1] We will prove the Mean Value theorem in Chapter 12.

$$\begin{aligned}
&|x_m - x_n| \\
=\ &|x_m - L + L - x_n| \\
\le\ &|x_m - L| + |L - x_n| \\
<\ &\varepsilon/2 + \varepsilon/2 \\
=\ &\varepsilon. \blacksquare
\end{aligned}$$

Theorem 10.13 is not very deep, but its converse is very much so. We will approach this through two lemmas describing properties of Cauchy sequences. Keep in mind that this is an "after the fact" organization of the proof, and the usefulness of each lemma may not be immediately apparent.

LEMMA 10.14: *If any subsequence of a Cauchy sequence converges, the sequence does also (and, by Theorem 9.18, to the same limit).*

PROOF: This proof is as easy to see in words as in symbols: The terms in a Cauchy sequence get closer and closer *together*, and if it has a convergent subsequence, *some* of its terms get close to a limit, and so *all* its terms must get close to that limit. All we need to do is translate this into a precise argument: Let (x_{n_k}) be a subsequence of (x_n) converging to L. Let $\varepsilon > 0$ be given and let N_1 be such that $|x_m - x_n| < \varepsilon/2$ whenever $m \ge n > N_1$. Let N_2 be such that $|x_{n_k} - L| < \varepsilon/2$ whenever $k > N_2$ and let $N = \max\{N_1, N_2\}$. If $k > N$, then

$$\begin{aligned}
&|x_k - L| \\
=\ &|x_k - x_{n_k} + x_{n_k} - L| \\
\le\ &|x_k - x_{n_k}| + |x_{n_k} - L| \\
\le\ &\varepsilon/2 + \varepsilon/2 \\
=\ &\varepsilon.
\end{aligned}$$

So $\lim x_n = L$. The estimate on $|x_k - x_{n_k}|$ holds because $n_k \ge k > N$. ■

We need to know whether a Cauchy sequence necessarily has a convergent subsequence. This is a consequence of the following, whose proof should have a familiar ring to it. Compare this to our earlier proof that a *convergent* sequence is bounded.

LEMMA 10.15: *A Cauchy sequence is bounded.*

PROOF: Let N be such that $|x_m - x_n| < 1$ whenever $m, n > N$.

Then we have, in particular, that $|x_{N+1} - x_n| < 1$ whenever $n > N$. Then $\{x_n\}$ is bounded above by $\max\{x_{N+1} + 1, x_1, \ldots, x_N\}$ and below by $\min\{x_{N+1} - 1, x_1, \ldots, x_N\}$. ■

The pieces are now in place, and we can establish the next part of the Big Theorem. Note that the Archimedean property (as stated in part (c) of the Big Theorem) implies itself (as stated in part (e) of the Big Theorem).

THEOREM 10.16: *If* **F** *is an ordered field in which the Bolzano-Weierstrass theorem holds, then a Cauchy sequence of elements of* **F** *converges to an element of* **F**.

PROOF: By Lemma 10.15, a Cauchy sequence is bounded. By the Bolzano-Weierstrass theorem for sequences, it has a convergent subsequence whose limit is an element of **F**. By Lemma 10.14, this is the limit of the sequence. ■

EXAMPLES 10.6: 1. Consider the decimal expansion $0.d_1d_2d_3\ldots$, and let $x_n = 0.d_1d_2\ldots d_n$. If $m > n$, then $x_m - x_n = 0.0\ldots 0d_{n+1}\ldots d_m < 1/10^n$. Thus (x_n) is a Cauchy sequence, and so it converges. This is another proof that every decimal expansion corresponds to a real number.

EXERCISES 10.6

1. Show directly that $\left(\frac{n+1}{n}\right)$ is a Cauchy sequence. (This time "directly" means do *not* argue that since this sequence converges it must be a Cauchy sequence.)

2. (a) If (a_n) and (b_n) are Cauchy sequences, show directly that (a_n+b_n) and (a_nb_n) are Cauchy sequences.

   (b) If (a_n) and (b_n) are Cauchy sequences, is (a_n/b_n) necessarily a Cauchy sequence?

   (c) If (a_n) and (b_n) are Cauchy sequences, give conditions on (a_n) and/or (b_n) that would guarantee that (a_n/b_n) is a Cauchy sequence.

3. (a) Show that a Cauchy sequence of integers must be eventually constant.

   (b) Complete and prove: If S is a set with the property ____________, then any Cauchy sequence of elements of S is eventually constant.

4. (a) Show that a Cauchy sequence can't have more than one sequential cluster point.

(b) Use Theorem 10.11 to prove Theorem 10.16.

5. (a) Suppose that (x_n) is a sequence with the property that there is a number $k < 1$ so that $|x_{n+1} - x_{n+2}| \leq k|x_n - x_{n+1}|$. Such a sequence is called **contractive**. Show that a contractive sequence converges.

   (b) Give an example of a convergent sequence that is *not* contractive.

   (c) If f is a differentiable function with the property that there is a number $k < 1$ such that $|f'(x)| \leq k$ for all x, show that the sequence given by $x_1 = a; x_{n+1} = f(x_n)$ is contractive (and hence converges).

   (d) Suppose the hypotheses of (c) are weakened to require only that $|f'(x)| \leq 1$ for all x. Does the result still hold?

   (e) With your calculator set for "radians," and starting with any number, press the "cos" key repeatedly. Can you explain this behavior?

6. Show that Theorem 9.16 fails for **Q**; that is, find a Cauchy sequence of rational numbers whose limit is not rational (since the terms in this sequence are also real numbers, the sequence must have a real limit).

7. Show directly that a bounded, monotone sequence is a Cauchy sequence. (Here "directly" means don't use this argument: A bounded monotone sequence converges. A convergent sequence is a Cauchy sequence.)

8. (a) Which of the combinations in the following chart are possible?

	BOUNDED		BOUNDED	
	UNBOUNDED		UNBOUNDED	
A	CONVERGENT	sequence	CONVERGENT	
(An)	DIVERGENT	with	DIVERGENT	
	MONOTONE	a(an)	MONOTONE	subsequence
	NOT MONOTONE		NOT MONOTONE	
	CAUCHY		CAUCHY	
	NOT CAUCHY		NOT CAUCHY	

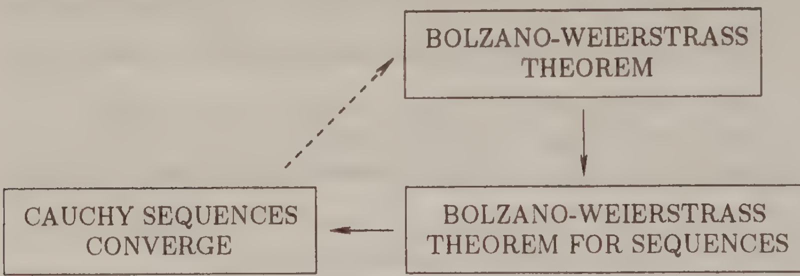
   (b) Which, if any, of the above combinations *must* happen?

   (c) Are there any combinations above that *can't* happen?

9. Why is the Archimedean property stated explicitly in part (e) of the Big Theorem but not in Theorem 10.16?

10.7 CLOSING THE LOOP

Our proofs of the results in the lower left corner of the Big Picture so far include the solid arrows in this diagram:



We will now fill in the dashed arrow. Recall that part (e) of the Big Theorem says "$\mathbf{F}$ has the Archimedean property, and a sequence in $\mathbf{F}$ converges to an element of $\mathbf{F}$ if and only if it is a Cauchy sequence," and part (c) says "$\mathbf{F}$ has the Archimedean property, and every bounded, infinite subset of $\mathbf{F}$ has a cluster point." The Archimedean property, again, implies itself, and there is nothing more to prove there. We will show that "Cauchy sequences converge" implies the Bolzano-Weierstrass theorem. More precisely:

THEOREM 10.17: *If $\mathbf{F}$ is an ordered field in which every Cauchy sequence converges to an element of $\mathbf{F}$, then every bounded, infinite subset of $\mathbf{F}$ has a cluster point that is an element of $\mathbf{F}$.*

PROOF: Let S be a bounded, infinite set. We must construct a Cauchy sequence converging to a cluster point of S. The proof of Theorem 7.5 gives us an idea of how to find a cluster point, and this proof is a modification of that one. Say $S \subseteq [a, b]$. Let $S_0 = S$, $I_0 = [a, b]$, and $x_0 \in S_0$. Let I_1 be the left half of I_0 if its intersection with S_0 is infinite, and the right half if it is not, and let $S_1 = S \cap I_1$. Then S_1 is infinite and so contains a point, x_1, different from x_0 (x_0 may not be in S_1, but $x_1 \in S_0$). We continue in this way (it is a familiar argument), producing sets $S_0, S_1, \ldots$, intervals $I_0, I_1, \ldots$, and points $x_0, x_1, \ldots$, with the following properties:

(i) I_n is either the right or left half of I_{n-1}.
(ii) $x_n \in S_k$ for all $n \geq k$.
(iii) $S_n \subseteq I_n$ for all n.
and (iv) $x_n \neq x_m$ for all $m \neq n$.

Only statement (ii) is not immediate. You will prove it in Exercise 10.7.1.

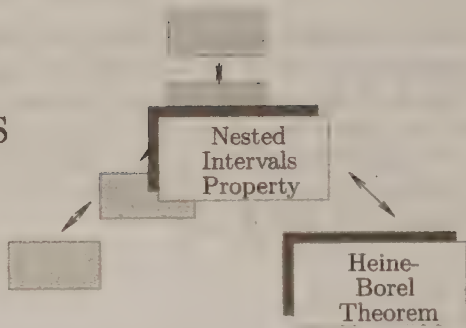
Now (i) implies that the infimum of the lengths of the I_n's is 0. Let $\varepsilon > 0$ be given and let N be such that the length of I_N is less than ε. If $m, n > N$, we have, by (ii) and (iii), that x_m and x_n are both elements of I_N, and so $|x_m - x_n| < \varepsilon$. Then (x_n) is a Cauchy sequence and so converges. Call its limit c. By (iv), (x_n) is not eventually constant. By Theorem 9.13, c is a cluster point of $\{x_n\}$, and since $\{x_n\} \subseteq S$, c is a cluster point of S. ∎

EXERCISES 10.7

1. Prove statement (ii) in the proof of Theorem 10.17.

Chapter 11

Compact Sets



11.1 THE EXTREME VALUE THEOREM

The theoretical side of calculus is mainly a study of continuous functions whose domains are closed, bounded intervals. We usually think of the most important results (the Intermediate Value theorem and the Extreme Value theorem[1]) as statements about functions, but on a deeper level they have just as much to say about the structure of intervals. In this chapter we will examine the property of closed, bounded intervals that makes the Extreme Value theorem work. We will discuss the Intermediate Value theorem in the next chapter. The Extreme Value theorem says:

If f is a continuous function whose domain is a closed, bounded interval, then f assumes a maximum on its domain.

"Assumes a maximum" means there is an element of the domain of f, say c, with $f(c) \geq f(x)$ for all x in the domain of f. This is the same as saying $\sup f(S) \in f(S)$. Notice, though, that this is *different* from saying that the range of f is bounded. The function $f(x) = x$ with domain $(0, 1)$, is bounded (we say a function is **bounded** if its range is bounded), but since $\sup f((0, 1)) = 1 \notin f(0, 1)$, f does not assume a maximum. Remember the distinction between "maximum" and "supremum."

This simple example also shows that the Extreme Value theorem does not hold if the domain of the function is an *open* interval and suggests there might be something more at stake here than just the behavior of functions. What is special about *closed, bounded* intervals? We can get insight into this if we think of other types of sets where the Extreme Value theorem holds. We want a theorem that says:

If f is a continuous function whose domain is ?????
then f assumes a maximum on its domain.

---

[1] There is a discussion of *why* these are the most important results in calculus at the end of the next chapter.

We see that we can put the words "a finite set" in place of the question marks. *Any* function whose domain is finite is continuous (Exercise 8.7.4). The range of such a function is also a finite set, and so it has a largest element. Closed, bounded intervals and finite sets share this property, which we give a name.

DEFINITION 11.1: The set $K \subseteq \mathbf{R}$ is **compact** if every continuous function $f : K \to \mathbf{R}$ assumes a maximum.

EXAMPLES 11.1: 1. The Extreme Value theorem says that closed, bounded intervals are compact, but *we haven't proved this yet* (it was "beyond the scope" of your calculus course). The proof of the Extreme Value theorem is one of the main goals of this chapter.

2. Finite sets are compact (we *have* proved this).

3. The whole real line is not compact. The function $f(x) = x$ is continuous but isn't bounded, and so it can't assume a maximum. We can use this same function, along with the Archimedean property, to show that $\mathbf{N}$ is not compact. Compactness is a property of the "large scale" structure of a set, as opposed to cluster points, for instance, which describe a set's "small scale" structure. The set of natural numbers certainly looks like a finite set when viewed on a small scale, but it is not compact.

4. Let $H = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. Then H is not compact. We can show this by finding one continuous function on H that doesn't assume a maximum. Note that $f(x) = 1/x$ is continuous on H (f is continuous on any set that doesn't contain 0), but f is not bounded on H (in fact, $f(H) = \mathbf{N}$).

5. Let $S = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. Even though S differs only slightly from the set H in the previous example, S *is* compact. Let f be a continuous function on S. If $f(0) = \sup f(S)$, we're done (f assumes a maximum at $x = 0$). Otherwise, let n_0 be such that $f(1/n_0) > f(0)$, and let $\varepsilon = f(1/n_0) - f(0) > 0$. The set $B = \{x \in S : f(x) \geq f(0) + \varepsilon\}$ is finite, since f is continuous at 0, and B is not empty since $1/n_0 \in B$. The maximum of f on S is just its maximum on B, which is attained for an element of B. You will check these statements in Exercise 11.1.2.

It is natural to ask how compactness is related to the usual set operations and the properties of sets we already know. We find right off the bat that the proofs of some of these results are very straightforward, while others are much more elusive.

THEOREM 11.2: *If A and B are compact, then $A \cup B$ is compact.*

PROOF: Let $f : A \cup B \to \mathbf{R}$ be continuous. We wish to show that f assumes a maximum on $A \cup B$. Since f is continuous on $A \cup B$, it is continuous on A and on B (Exercise 8.7.3). Since A is compact, f assumes a maximum on A, say $M_A = f(a)$. Likewise, f assumes a maximum on B, say $M_B = f(b)$. The maximum value of f on $A \cup B$ is $\max\{M_A, M_B\}$, which is assumed at either a or b. ∎

We now may use induction to prove:

COROLLARY 11.3: *The union of any finite collection of compact sets is compact.*

PROOF: Left as Exercise 11.1.4. ∎

It is quite a bit more difficult to prove in this way that the intersection of two compact sets is compact since a function can be continuous on an intersection of two sets without being continuous on either set. We will prove that intersections of compact sets are compact when we have more techniques at our disposal. (When we get to it, the proof will be one line long—an illustration of the value of waiting for the right tool to come along.)

Compactness is always a bit of a mystery. Perhaps we can relate it to more familiar ideas. Example 11.1.4 works as it does because the set H has a cluster point that it does not contain. In other words, H is *not closed*. Example 11.1.3, on the other hand, works because the sets involved are not bounded. This leads us to suspect:

THEOREM 11.4: *A compact set is closed and bounded.*

PROOF: This theorem is of the form $A \Rightarrow (B \text{ and } C)$, and so we must prove both $A \Rightarrow B$ and $A \Rightarrow C$. We will do both by the contrapositive. First, suppose the set S is not bounded. The function $f(x) = |x|$ is continuous on the whole real line, and therefore it is continuous on any subset of the real line. But if S is not bounded, f is not bounded on S, and so S isn't compact. Now suppose the set T is not closed. Then it has a cluster point, say t, that it does not contain. Let $f(x) = 1/|x-t|$. Then f is continuous everywhere except at t. But $t \notin T$, and so f is continuous on T. Since t is a cluster point of T, there are elements of T as close as we like to t. These may be chosen to give values of f as large as we like. Thus f is not bounded on T, and since such a function exists, T is not compact (compare these proofs with Examples 11.1.3 and 11.1.4). ∎

EXERCISES 11.1

1. Give an example of a closed, bounded interval and a (discontinuous) function defined on it that assumes no maximum value.

2. Prove the statements made in Example 11.1.5 (see also Exercise 5.1.7).

3. Prove in detail that every function whose domain is a finite ~~set~~ is continuous. subset of the real number line

4. Prove Corollary 11.3.

5. (a) Make the argument in the second part of the proof of Theorem 11.4 more precise (the proof as stated says "...we can do this ...we can do that ...")

   (b) Restate the proof of Theorem 11.4 so that it is done *directly* (not by the contrapositive).

6. (a) Show that the union of an infinite collection of compact sets need not be compact.

   (b) Show that it is possible for the union of an infinite collection of compact sets to be compact.

7. Show that the definition of "compact" may be restated with "maximum" replaced by "minimum."

8. Show that the *difference* of two compact sets need not be compact.

9. Consider whether the definition of compactness is equivalent to saying just that every continuous function defined on a set is *bounded*.

10. Notice that, in the definition of compactness, the set K need only be a subset of a topological space (but not necessarily $\mathbf{R}$).

    (a) If X is an infinite set with the discrete topology, show that X is not compact. (This is a bit of a trick question; be careful.)

    (b) If X is any set with the indiscrete topology, show that X *is* compact.

11.2 THE COVERING PROPERTY

The converse of Theorem 11.4 would be very useful. Deciding whether a set is closed and bounded would seem far easier than deciding whether it is compact. Our solution to this problem will lead us into some very abstract mathematics. Being abstract often means looking at a problem

topologically, that is, finding a way to express an idea in terms of open sets. The definition of compactness is only partly topological. It involves the (topological) issue of continuous functions, but we must also consider the (nontopological) question of the ordering of the real line to make sense of "maximums."

The Covering property is a tricky concept. To get an idea what it is about, let us consider two sets we know to be compact (a finite set and the set S in Example 11.1.5) and see if they have anything else in common. Our claim that "Every function whose domain is a finite set is continuous" is based on bits and pieces of other proofs. If we were to assemble a detailed proof of this statement, we would see that it hinges on the fact that we can enclose a finite set in a collection of open intervals. (That every function on such a set is continuous then follows because each point of the set is open.)

Suppose we *suspect* that we can cover a set with intervals like this, but we don't know exactly where these intervals are, how big they ought to be, or how many of them we need. In our desire to be sure the set is covered, we might get carried away and bury the set in intervals, like this (!):

Now among these intervals (no matter how many there are and even if there are infinitely many), there is at least one that contains a_1. We can pick one of these out and label it I_1. Likewise, there is an interval that contains a_2. We can pick such an interval out and label it I_2 (it may even be that a_2 is in I_1). We can continue in this way and find a finite collection of intervals, chosen from the original bunch, that contains the whole set (here there will be no more than five intervals chosen).

What happens if we try this with the set S in Example 11.1.5 (which is compact)? If we cover S with open intervals, at least one of them must contain 0. By Corollary 6.2.b, this particular interval must also contain another element of S, and in fact must contain *all but finitely many* of the elements of S (be sure you see why this is so). Since only finitely many points of S remain uncovered, we can pick out finitely many more intervals containing the remaining elements of S as in the previous example.

In contrast, consider the set of natural numbers, which we know is not compact. Let $I_n = (n - 1/2, n + 1/2)$ for $n = 1, 2, \ldots$. These intervals cover $\mathbf{N}$, but if we remove *even a single one*, those that remain will no longer cover $\mathbf{N}$ (if we remove I_{237}, for instance, $n = 237$ will no longer be

contained in any of the intervals). Certainly no finite collection of these intervals can cover $\mathbf{N}$. We have found a property, having something to do with "covering" sets with open intervals, that is shared by the two sets we know are compact but not possessed by a set we know is not compact. To make our definition purely topological, we replace "open intervals" with "open sets":

DEFINITION 11.5: A collection of open sets $\{U_\alpha : \alpha \in \mathcal{A}\}$ is an **open cover** of the set S if $S \subseteq \bigcup_{\alpha \in \mathcal{A}} U_\alpha$. If $\{U_\alpha\}$ is an open cover of S, we say that we have **covered** S with $\{U_\alpha\}$.

Remember that in using α as a subscript, we are making no commitment as to the cardinality of the index set $\mathcal{A}$. If we were to write $\{U_n\}$ or $\{U_k\}$, we might think that $\mathcal{A}$ has to be countable, which may not be the case.

Let us put our observation about finite sets in terms of Definition 11.5. Let $A = \{a_1, a_2, \ldots, a_n\}$ and suppose $\{U_\alpha\}$ is an open cover of A. Since $a_1 \in A$, it must be that a_1 is in one of the sets U_α. Call this set U_{α_1}. Similarly, we may find sets $U_{\alpha_2}, \ldots, U_{\alpha_n}$ in $\{U_\alpha\}$ containing $a_2, \ldots, a_n$, respectively. *Even though* $\{U_\alpha\}$ *may have infinitely many sets in it, we need only finitely many of them to cover* A. The collection $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ is called a **subcover** (this particular one is a **finite subcover**). It is extremely important to recognize what has happened here. We have **NOT** said:

↓ NO ↓ NO ↓ NO ↓ NO ↓

We can cover A with finitely many open sets.

↑ NO ↑ NO ↑ NO ↑ NO ↑

But we **HAVE** said:

↓ YES ↓ YES ↓ YES ↓ YES ↓

We can cover A with finitely many sets *chosen from* $\{U_\alpha\}$.

↑ YES ↑ YES ↑ YES ↑ YES ↑

At first glance, these statements may not seem very different. Upon reflection, though, we find that the first of them doesn't say anything useful at all. The whole real line is an open set, and so we can cover *anything* with finitely many open sets (we can do it with just one: the whole real line). Producing a finite open cover is not a challenge. It becomes one only when we must cover our set with finitely many sets chosen from a previously specified collection.

DEFINITION 11.6: A set **has the covering property** if any open cover of it has a finite subcover.

Note that "finite" here refers to the number of sets in the subcover, not to the cardinalities of the individual sets.

EXERCISES 11.2

1. Show directly that the union of two sets having the covering property has the covering property.

2. In the discussion preceding the definition of an open cover, why is it necessary to replace "open intervals" with "open sets" to make the definition purely topological?

3. (a) Show that a closed, bounded set having exactly one cluster point has the covering property.

   (b) Show that "closed" and "bounded" are both necessary to make the statement in (a) true.

   (c) Show that a closed, bounded set having finitely many cluster points has the covering property.

   (d) In (c) you showed that if S is closed and bounded and S' is finite, then S has the covering property. Now show that if S is closed and bounded and S'' is finite, then S has the covering property.

4. Show that a set with the covering property is compact in this way: If the set is not compact, there is a continuous function f defined on it that attains no maximum. Hence, if $f(x)$ is any value of f, there is a number y in the set with $f(y) > f(x)$. Construct an open cover of the set with no finite subcover.

5. (a) Show that a set of the form $[a, \infty)$ is not compact by displaying a continuous function on it having no maximum.

   (b) Show that a set of the form $[a, \infty)$ does not have the covering property by displaying an open cover of it having no finite subcover.

6. (a) Show that the set H in Example 11.1.4 does not have the covering property.

   (b) Supply the details of the proof that the set S in Example 11.1.5 has the covering property.

7. (a) Suppose S is an infinite set having the covering property and that $\{U_\alpha\}$ is an open cover of S. Show that there is an α^* so that $U_{\alpha^*} \cap S$

is infinite.

(b) Give an example of a set and a cover, as in (a), such that there is only one such set U_{α^*}.

(c) Give an example of a set S and an open cover $\{U_\alpha\}$ such that $U_\alpha \cap S$ is infinite for all α, but S does not have the covering property.

8. (a) Show that an infinite set with the discrete topology fails to have the covering property.

   (b) Show that any set with the indiscrete topology has the covering property.

9. Consider the set consisting of the real numbers and another symbol, say ∞. This set is denoted $\mathbf{R} \cup \{\infty\}$. We say that a subset of $\mathbf{R} \cup \{\infty\}$ is a **neighborhood of** ∞ if it contains the complement of a compact set. Neighborhoods of the other elements of the set are defined in the usual way.

   (a) Show that a subset of $\mathbf{R} \cup \{\infty\}$ that does not contain ∞ is open if and only if it is an open subset of $\mathbf{R}$ (in the usual sense).

   (b) State and prove a condition that says when a subset of $\mathbf{R} \cup \{\infty\}$ that does contain ∞ is open.

   (c) Show that the open subsets of $\mathbf{R} \cup \{\infty\}$ form a topology.

   (d) Show that a function $f : \mathbf{R} \cup \{\infty\} \to \mathbf{R}$ is continuous if (i) it is continuous on $\mathbf{R}$ in the usual sense, and (ii) $f(\infty) = \lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x)$ (where both of these limits are taken in the sense of ordinary calculus).

   (e) Show that $f : \mathbf{R} \cup \{\infty\} \to \mathbf{R} \cup \{\infty\}$ given by

$$f(x) = \begin{cases} 1/x & x \neq 0 \\ \infty & x = 0 \\ 0 & x = \infty \end{cases}$$

   is continuous, while $g : \mathbf{R} \cup \{\infty\} \to \mathbf{R} \cup \{\infty\}$ given by

$$g(x) = \begin{cases} e^x & x \neq \infty \\ \infty & x = \infty \end{cases}$$

   is not.

   (f) Show that $\mathbf{R} \cup \{\infty\}$ is compact.

   (g) Show that $\mathbf{R} \cup \{\infty\}$ has the covering property.

   (h) The set $\mathbf{R} \cup \{\infty\}$ is called the **one-point compactification** of $\mathbf{R}$. Show that this procedure works for any noncompact topological space. That is, if X is such a space, consider the set $X \cup \{\infty\}$, where a subset is considered a neighborhood of ∞ if it contains the complement of a

compact subset of X. Show that $X \cup \{\infty\}$ is a compact topological space.

(i) What happens if you construct the one-point compactification of a set that is already compact?

10. Now consider the real numbers together with *two* new symbols, say $+\infty$ and $-\infty$ (these are just two symbols, the signs don't indicate arithmetic operations). We will say that a set is a **neighborhood of** $+\infty$ if it contains the complement of a closed set that is bounded above, and that a set is a **neighborhood of** $-\infty$ if it contains the complement of a closed set that is bounded below.

(a) Show that a subset of $\mathbf{R} \cup \{-\infty, +\infty\}$ that does not contain $-\infty$ or $+\infty$ is open if and only if it is an open subset of $\mathbf{R}$ (in the usual sense).

(b) State and prove a condition that determines whether a subset of $\mathbf{R} \cup \{-\infty, +\infty\}$ that *does* contain $-\infty$ or $+\infty$ is open.

(c) Show that the open subsets of $\mathbf{R} \cup \{-\infty, +\infty\}$ form a topology.

(d) Show that a function $f : \mathbf{R} \cup \{-\infty, +\infty\} \to \mathbf{R}$ is continuous if (i) it is continuous on $\mathbf{R}$ in the usual sense, (ii) $f(-\infty) = \lim_{x\to-\infty} f(x)$, and (iii) $f(\infty) = \lim_{x\to\infty} f(x)$.

(e) Show that $g : \mathbf{R} \cup \{-\infty, +\infty\} \to \mathbf{R} \cup \{-\infty, +\infty\}$ given by

$$g(x) = \begin{cases} e^x & x \neq -\infty, \infty \\ \infty & x = \infty \\ 0 & x = -\infty \end{cases}$$

is continuous.

(f) Can the function $f : \mathbf{R} \cup \{-\infty, +\infty\} \to \mathbf{R} \cup \{-\infty, +\infty\}$ given by $f(x) = \dfrac{1}{x}$ for $x \in \mathbf{R}$ be defined at $0, -\infty$, and $+\infty$ in such a way as to make it continuous?

(g) Show that $\mathbf{R} \cup \{-\infty, +\infty\}$ is compact.

(h) Show that $\mathbf{R} \cup \{-\infty, +\infty\}$ has the covering property.

(i) $\mathbf{R} \cup \{-\infty, +\infty\}$ is the **two-point compactification** of $\mathbf{R}$. This procedure does not make sense in the general topological setting. Why?

(j) Show that any rational function on the real numbers can be extended in such a way that it is continuous as a function from $\mathbf{R} \cup \{-\infty, \infty\}$ to $\mathbf{R} \cup \{-\infty, \infty\}$.

11.3 THE HEINE-BOREL THEOREM

In our last example above, we saw that the set of natural numbers does not have the covering property. Here is another example: Let $I = (0, 1)$ and let $U_n = (1/n, 2)$ for $n = 1, 2, \ldots$. Then $I \subseteq \bigcup_n U_n$, but no finite collection of the sets U_n covers I (be sure you see why this is so). In these examples we can see the essence of the theorem we seek, called the **Heine-Borel Theorem**, which says: *A set is closed and bounded if and only if it has the covering property.* More precisely, we will show:

THEOREM 11.7: *If* $\mathbf{F}$ *is an Archimedean ordered field having the Nested Intervals property, then the following is also true: A subset of* $\mathbf{F}$ *is closed and bounded if and only if it has the covering property.*

This theorem has the most complicated logical structure of any we have seen, and so we will examine it closely before we begin the proof. The theorem is of the form $(A \Rightarrow B) \Rightarrow (C \Leftrightarrow D)$, where

A : $\{I_n\}$ is a nest of closed, bounded intervals

B : $\bigcap_n I_n \neq \emptyset$

C : The set S is closed and bounded

D : S has the covering property

Since there is an "and" in the conclusion (where?), the theorem has two parts:

$$(A \Rightarrow B) \Rightarrow (D \Rightarrow C) \text{ and } (A \Rightarrow B) \Rightarrow (C \Rightarrow D),$$

which (by Exercise 1.4.6.a) are in turn equivalent to

$$((A \Rightarrow B) \text{ and } D) \Rightarrow C \text{ and } ((A \Rightarrow B) \text{ and } C) \Rightarrow D.$$

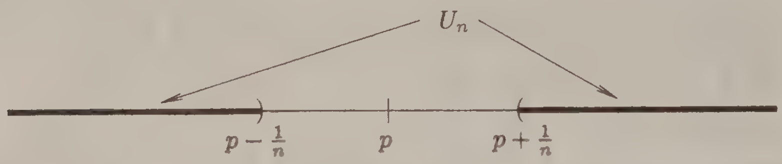
We must prove: (1) If the Nested Intervals property holds $(A \Rightarrow B)$ and S has the covering property (D), then S is closed and bounded (C). (This one is easy.)

(2) If the Nested Intervals property holds and the set S is closed and bounded, then S has the covering property. (This is much more significant.)

PROOFS: (1) Suppose that S has the covering property and let $U_n = (-n, n)$ for $n \in \mathbf{N}$. Then $\{U_n\}$ is a cover of S (it is a cover of the whole line), and since S has the covering property some finite collection of these sets must cover S. The union of any finite collection of the sets U_n is bounded (the union is just the one with the largest subscript), and

so S is contained in a bounded set and S is bounded.

Now suppose S has the covering property and $p \notin S$. We will show that p is not a cluster point of S (this will tell us that S is closed). Let $U_n = \mathbf{R}\backslash[p - 1/n, p + 1/n]$ for $n \in \mathbf{N}$. Then U_n is open for each n and $\bigcup_n U_n = \mathbf{R}\backslash\{p\}$ (why?).



Since $p \notin S$, $\{U_n\}$ is a cover of S. Since S has the covering property, there is a finite collection of the sets $\{U_n\}$ that covers S. One of the sets in this collection will have the largest subscript, call it n_0. The union of this finite collection contains S but contains no element of $[p-1/n_0, p+1/n_0]$. Hence $(p - 1/n_0, p + 1/n_0) \cap S = \emptyset$, and p is not a cluster point of S.

(2) This classic proof proceeds by contradiction. Suppose S is a closed, bounded set and that $\{U_\alpha\}$ is an open cover of S having no finite subcover. We will say that a subset T of S has "property $\mathcal{B}$" (for "bad") if no finite subcollection of $\{U_\alpha\}$ covers T. Note that S itself has $\mathcal{B}$. It is important to understand property $\mathcal{B}$ clearly. In order for T to have $\mathcal{B}$, it must be impossible to cover T with finitely many sets *chosen from* $\{U_\alpha\}$.

Since S is bounded, it may be contained in a closed, bounded interval, say I_0. Let $S_0 = S = I_0 \cap S$. Divide I_0 in the middle and call the left half I_L and the right half I_R. One or the other (or both) of $S_0 \cap I_L$ or $S_0 \cap I_R$ must have $\mathcal{B}$ (if neither of them had $\mathcal{B}$, then S_0 would not have $\mathcal{B}$). Let I_1 be I_L if $S_0 \cap I_L$ has $\mathcal{B}$ and I_R otherwise, and let $S_1 = I_1 \cap S$. Everything we have said about S_0 is true of S_1, and so we may repeat this process to obtain sets $S_2, S_3, \ldots$, and a nest of closed, bounded intervals $\{I_n\}$. The length of each interval I_n is half that of the previous one, and so the infimum of their lengths is 0. By the Nested Intervals property, there is a real number s so that $\bigcap_n I_n = \{s\}$. Now s is a cluster point of S (this follows as in the proof of the Bolzano-Weierstrass theorem—notice that a set with property $\mathcal{B}$ must be infinite). Since S is closed, we have $s \in S$. Thus there is an element of the cover, say U_{α^*}, with $s \in U_{\alpha^*}$. Now U_{α^*} is open, and so there is an $\varepsilon > 0$ with $(s - \varepsilon, s + \varepsilon) \subseteq U_{\alpha^*}$. The infimum of the lengths of the intervals I_n is 0, and so there is one, say I_{n^*}, with length less than ε (in fact, all but finitely many of them have this property). Then $S_{n^*} \subseteq I_{n^*} \subseteq (s - \varepsilon, s + \varepsilon) \subseteq U_{\alpha^*}$, contradicting the way the sets S_n were chosen (S_{n^*} fails to have property $\mathcal{B}$ in a big way since it can be covered by *one* set from $\{U_\alpha\}$). ∎

So far we have proved implications like this:

$$\begin{array}{c} S \text{ is compact} \\ \Downarrow \\ S \text{ is closed and bounded} \\ \Downarrow\ \Uparrow \\ S \text{ has the covering property} \end{array}$$

(with the bottom $\Downarrow$ depending on the Nested Intervals property). Now we will show that the various properties are equivalent by showing that the covering property implies compactness.

We will first prove something called a "preservation theorem." A preservation theorem is one that says "*If a set has* This *property and you do* That *to it, the result will also have* This *property*" (This is preserved under That).

THEOREM 11.8: *If the set S has the covering property and $f : S \to \mathbf{R}$ is continuous, then $f(S)$ has the covering property.*

PROOF: We will use the forward-backward method (in this proof there is a lot of "forward" and not much "backward"). It is $f(S)$ we wish to show has the covering property, and so our proof should begin and end like this:

▽▽▽

Let $\{U_\alpha\}$ be an open cover of $f(S)$.

⋆ ⋆ ⋆

Then $\{U_{\alpha_k} : k = 1, \ldots, m\}$ is a cover of $f(S)$.

△△△

We now have *open sets* in the *range* of the *continuous function* f. This collection of words together suggests strongly that we apply the definition of continuity:

▽▽▽

Let $\{U_\alpha\}$ be an open cover of $f(S)$.

$\hookrightarrow$ Consider the open sets $\{f^{-1}(U_\alpha)\}$.

⋆ ⋆ ⋆

Then $\{U_{\alpha_k} : k = 1, \ldots, m\}$ is a cover of $f(S)$.

△△△

All we know about S is that it is contained in the domain of f and has the covering property. Now $\{f^{-1}(U_\alpha)\}$ is a collection of open sets contained in the domain of f. We should check whether $\{f^{-1}(U_\alpha)\}$ is a cover of S.

▽▽

Let $\{U_\alpha\}$ be an open cover of $f(S)$.

Consider the open sets $\{f^{-1}(U_\alpha)\}$.

$\hookrightarrow$ Let $s \in S$.

$\hookrightarrow$ Then $f(s) \in f(S)$ [by definition of $f(S)$].

$\hookrightarrow$ So $f(s) \in U_{\alpha_0}$ for some α_0 [because $\{U_\alpha\}$ is a cover of $f(S)$] and $s \in f^{-1}(U_{\alpha_0})$ [by definition of $f^{-1}(U_{\alpha_0}]$.

$\hookrightarrow$ Hence $\{f^{-1}(U_\alpha)\}$ is an open cover of S.

★ ★ ★

Then $\{U_{\alpha_k} : k = 1, \ldots, m\}$ is a cover of $f(S)$.

△△

The next step is now apparent:

▽▽

Let $\{U_\alpha\}$ be an open cover of $f(S)$.

Consider the open sets $\{f^{-1}(U_\alpha)\}$.

Let $s \in S$.

Then $f(s) \in f(S)$ [by definition of $f(S)$].

So $f(s) \in U_{\alpha_0}$ for some α_0 [because $\{U_\alpha\}$ is a cover of $f(S)$] and $s \in f^{-1}(U_{\alpha_0})$ [by definition of $f^1(U_{\alpha_0})$].

Hence $\{f^{-1}(U_\alpha)\}$ is an open cover of S.

$\hookrightarrow$ Since S has the covering property, there is a finite subcover $\{f^{-1}(U_{\alpha_1}), \ldots, f^{-1}(U_{\alpha_m})\}$.

★ ★ ★

Then $\{U_{\alpha_k} : k = 1, \ldots, m\}$ is a cover of $f(S)$.

△△

We hope that $\{U_{\alpha_k} : k = 1, \ldots, m\}$ is a cover of $f(S)$:

▽▽

Let $\{U_\alpha\}$ be an open cover of $f(S)$.

Consider the open sets $\{f^{-1}(U_\alpha)\}$.

Let $s \in S$.

Then $f(s) \in f(S)$ [by definition of $f(S)$].

So $f(s) \in U_{\alpha_0}$ for some α_0 [because $\{U_\alpha\}$ is a cover of $f(S)$] and $s \in f^{-1}(U_{\alpha_0})$ [by definition of $f^{-1}(U_{\alpha_0})$].

Hence $\{f^{-1}(U_\alpha)\}$ is an open cover of S.

Since S has the covering property, there is a finite subcover $\{f^{-1}(U_{\alpha_1}), \ldots, f^{-1}(U_{\alpha_m})\}$.

↪ Let $t \in f(S)$.

↪ There is an element s of S with $t = f(s)$.

↪ s is in one of the sets $f^{-1}(U_{\alpha_1}), \ldots,$ or $f^{-1}(U_{\alpha_m})$.

↪ So t is in the corresponding set $U_{\alpha_1}, \ldots,$ or U_{α_m}.

Then $\{U_{\alpha_k} : k = 1, \ldots, m\}$ is a cover of $f(S)$. ∎

△△

Several of the exercises in Section 11.2 hint at the following theorem.

THEOREM 11.9: *If S has the covering property, it is compact.*

PROOF: We must show that any continuous, real-valued function whose domain is S attains a maximum. Since S has the covering property, $f(S)$ has the covering property, and so $f(S)$ is closed and bounded. We revise a familiar argument: Since $f(S)$ is bounded, there exist a_0, b_0 such that $f(S) \subseteq [a_0, b_0] = I_0$. Divide I_0 in half in the middle. Let I_1 be the right half of I_0 if that half contains any element of $f(S)$, and the left half otherwise. Continue in this manner, producing a nest of closed, bounded intervals with the properties: (1) $I_n \cap S \neq \emptyset$ for all n, and (2) If I_n is the left half of I_{n-1}, the right half of I_{n-1} contains no element of $f(S)$. Let $\{u\} = \bigcap_n I_n$. You will show in Exercise 11.3.2 that $u = \sup S$.([2]) Since $f(S)$ is closed, it contains its supremum (Exercise 8.3.1); that is, $\sup f(S) \in f(S)$. Thus S is compact. ∎

It is now a simple thing to extend Theorem 11.2 to intersections:

COROLLARY 11.10: *The intersection of any collection of compact sets is compact.*

PROOF: The intersection of any collection of closed, bounded sets is closed and bounded. (This is a corollary to the whole circle of results, not just to Theorem 11.9.) ∎

We have three equivalent characterizations of compactness: the definition, the covering property, and "closed and bounded." We may use "compact" to refer to any one of them. *It is traditional to take the covering property*

---

[2] The previous few lines are an outline of the proof that the Nested Intervals property implies the Least Upper Bound property. We might have simply said "S is bounded above, and so it has a supremum," but we must do the proof as we have in view of the theorem's position in the Big Picture.

as the definition of compactness since it is more "topological." If we do this, the Heine-Borel theorem also assumes its traditional form:

A set is compact if and only if it is closed and bounded.

Since the covering property is equivalent to compactness, we have already established the following preservation theorem. The direct proof of it is still instructive in its simplicity. Be sure you see why this proof begins the way it does.

THEOREM 11.11: *If the set S is compact and $f : S \to \mathbf{R}$ is continuous, then $f(S)$ is compact.*

PROOF: Let $g : f(S) \to \mathbf{R}$ be continuous. Then $g \circ f : S \to \mathbf{R}$ is continuous and, since S is compact, $g \circ f$ attains a maximum value, say $g(f(s))$. Since f maps S onto $f(S)$, there are no inputs to g that are not of the form $f(x)$, and g can assume no value larger than $g(f(s))$. Thus g attains a maximum at $f(s)$, and $f(S)$ is compact. ∎

EXERCISES 11.3

1. Verify the statement in the proof of Theorem 11.7 that a set with property $\mathcal{B}$ must be infinite.

2. Show that the point u obtained in the proof of Theorem 11.9 is in fact $\sup S$. (In doing this, you will show directly that the Nested Intervals property implies the Least Upper Bound property.)

3. (a) Complete the proof of the example preceding Theorem 11.7.

   (b) Where is the Archimedean property used in the proof of the Heine-Borel theorem?

4. (a) Suppose S is closed and bounded and T is a closed subset of S. Show that T is closed and bounded.

   (b) Suppose S has the covering property and T is a closed subset of S. Show directly that T has the covering property.

   (c) Suppose S is compact and T is a closed subset of S. Show directly that T is compact. (This is very difficult.) Parts (a), (b), and (c) all say the same thing. Which was easiest?

5. (a) Construct an example of an open cover of the whole real line that has *uncountably* many sets in it and has no finite subcover.

   (b) Show that, for any natural number n, the interval $[-n, n]$ can be covered by finitely many of the sets you found in (a).

(c) Show that the real line can be covered by *countably* many of the sets you found in (a).

(d) Show that the result of (c) is true in general: Any open cover of $\mathbf{R}$ can be reduced to a *countable* subcover. [This is not quite as useful as if $\mathbf{R}$ were compact, but it does say something (however mysterious) about its topology. A topological space with the property that any open cover can be reduced to a countable subcover is called a **Lindelöf space.**]

6. (a) Show that the set made up of the range of a convergent sequence together with its limit is closed and bounded.

   (b) Show directly that the set made up of the range of a convergent sequence together with its limit has the covering property.

   (c) Give an example of a divergent sequence whose range is compact.

7. Suppose K is compact, $\{U_\alpha\}$ is an open cover of K with finite subcover $\{U_{\alpha_n}\}$, and $x \in K$. Let $\varepsilon_x = \sup\{\varepsilon : (x-\varepsilon, x+\varepsilon) \subseteq U_{\alpha_n} \text{ for some } n\}$.

   (a) Show that $\varepsilon_x > 0$ for all $x \in K$.

   (b) Argue that the function $f : K \to \mathbf{R}$ defined by $f(x) = \varepsilon_x$ is continuous.

   (c) Show that $f(x)$ assumes a positive minimum on K.

   (d) Show that (c) can interpreted in this way: *For any open cover of a compact set K, there is a positive number δ so that any interval of length less than δ and containing a point of K is contained in a single element of the cover.* This result is called the **Lebesgue Number Lemma**, and δ is called the **Lebesgue Number** of the cover. We have used a good bit of our knowledge of compact sets to do this proof. A direct proof is more difficult.

8. Show that a set is compact if and only if any sequence contained in it has a subsequence that converges to an element of the set.

9. Show that an analogue of the Nested Intervals property holds for compact sets, that is, if $K_1 \supseteq K_2 \supseteq \ldots$ is a nest of nonempty compact sets, then $\bigcap_n K_n \neq \emptyset$.

10. Show that an infinite compact set must have a cluster point that is in the set.

11. (a) Show that a nonempty compact set has and contains a supremum and infimum.

    (b) If your proof for (a) began with the phrase "Since the set is compact, it is closed and bounded ...," do another proof based only on

the definition of compactness. If you used the definition to do part (a), do another proof in which you begin by noting that a compact set is closed and bounded.

11.4 CLOSING THE LOOP

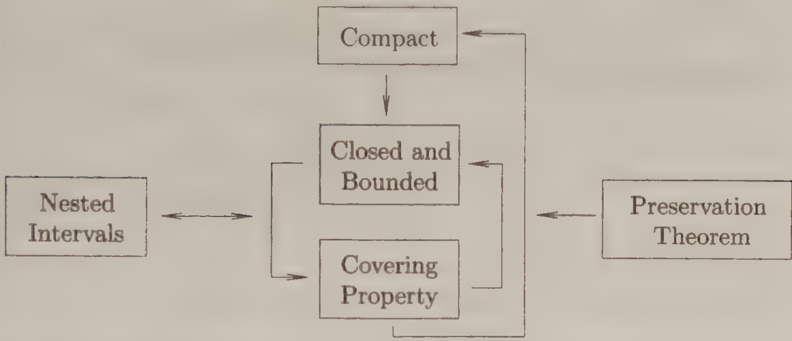
THEOREM 11.12: *An ordered field in which the Heine-Borel theorem holds also has the Archimedean property and the Nested Intervals property.*

PROOF: The Archimedean property is the same as the statement that $\mathbf{N}$ is unbounded. We have seen that $\mathbf{N}$ fails to have the covering property. By the Heine-Borel theorem, this means that $\mathbf{N}$ is either not closed or not bounded. But we know that $\mathbf{N}$ *is* closed, and so it must be that $\mathbf{N}$ is unbounded.

Now suppose the Nested Intervals property fails. Then there is a nest of closed, bounded intervals, $\{I_n\}$, with $\bigcap_n I_n = \emptyset$. Let $U_n = \mathbf{R}\backslash I_n$ for each n. Then $\{U_n\}$ is an open cover of the whole real line. (This is the only tricky part of the proof—the points *not* in $\bigcup_n U_n$ are the points *in* $\bigcap_n I_n$, a set we have assumed to be empty!) So $\{U_n\}$ is an open cover of, among other things, I_1, and I_1 is closed and bounded. Any finite subcollection of $\{U_n\}$ will have a set with the largest subscript, say n_0. The union of the finite collection is just U_{n_0}. The midpoint of I_{n_0}, an element of I_1, is not contained in U_{n_0}. Hence I_1 is not covered by any finite subcollection of $\{U_n\}$, and so it is closed and bounded but not compact, a contradiction. ∎

Here again we see that the negation of the conclusion gives us specific useful information, making the proof by contradiction a good bet. We did not use the full power of the Heine-Borel theorem here. We needed to know only that a closed, bounded *interval* has the covering property. You will show this, in a sense, in Exercise 11.4.2, but remember that in Exercise 11.3.7 you used the Heine-Borel theorem to obtain the Lebesgue number lemma. Here is a case where a very general result is easier to obtain than a specific one, a situation that is not as rare as one might expect. Theories are organized (in hindsight) to include only those ideas crucial to the discussion. Compactness is essentially topological, and to deal with intervals (and the attendant order structure) tends to obscure the real issues.

Finally, note that the lower right corner of the Big Picture is only an abbreviation for what has actually happened. Our proof has really gone like this:



EXERCISES 11.4

1. (a) Construct a closed, bounded subset of the field of formal rational functions that does not have the covering property.

   (b) If $\mathbf{F}$ is any non-Archimedean ordered field, must there necessarily be a closed, bounded subset of $\mathbf{F}$ that does not have the covering property?

2. (a) Find a closed, bounded subset S of $\mathbf{Q}$ and a function $f : S \to \mathbf{R}$ such that f is continuous but does not attain a maximum on S. (Note that a subset of $\mathbf{Q}$ is closed if it is *closed, that is, if it is the intersection of a closed set with $\mathbf{Q}$.)

   (b) Find a closed, bounded subset S of $\mathbf{Q}$ and an open cover of S with no finite subcover.

3. Recall the definition of "metric" given in Exercises 4.7.4.

   (a) Let $X = P(\mathbf{R})$, the power set of $\mathbf{R}$. For $A, B \in X$, we define the "distance" between two sets to be $d(A, B) = \inf\{|a - b| : a \in A, b \in B\}$. Is d a metric on X? Does d satisfy *any* of the conditions for being a metric?

   (b) Give a condition that will guarantee that $d(A, B) = 0$ whether or not $A = B$.

   (c) Show that the function $d(A, B)$ can be equal to 0 even if A and B are disjoint.

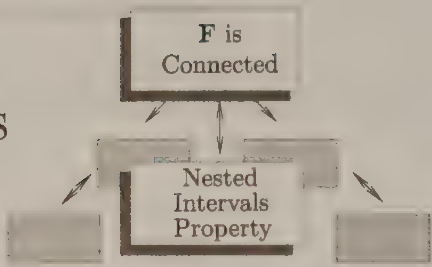
   (d) Show that if A and B are disjoint and compact, then $d(A, B) > 0$.

   (e) Is d a metric on the collection of compact sets?

4. Use the Lebesgue Number lemma to show that a closed, bounded interval has the covering property. (You *will* need the Lebesgue Number lemma to do this. If you think you have done a proof without it, examine your argument *very* carefully.)

Chapter 12

Connected Sets



12.1 THE INTERMEDIATE VALUE THEOREM

Consideration of the Extreme Value theorem led us in the last chapter to the idea of compactness. Now we will think about the Intermediate Value theorem in the same way. The Intermediate Value theorem says:

If f is a continuous, real-valued function defined on an interval, and f takes on a positive value at some point a and a negative value at some point b, then there is a point c between a and b where $f(c) = 0$.

(Sometimes a and b are required to be the endpoints of the interval, but it is not necessary. This form of the theorem is more useful for our purposes.)

Like the Extreme Value theorem, the Intermediate Value theorem has much to tell us about both intervals and functions. We will say that a function **has the intermediate value property** on a set if it takes on the value 0 somewhere in the set between any two points where its values have opposite signs. Evidently, this depends on both the function and the set. The Intermediate Value theorem says that any continuous function has the intermediate value property on any interval.

When we began our examination of the Extreme Value theorem, we looked for examples of sets other than intervals for which the conclusion of the theorem held. This is not a good way to begin discussion of the Intermediate Value property, for reasons that will soon be apparent. Instead, let's think of examples where the conclusion does *not* hold.

EXAMPLES 12.1: 1. Let $A = \{0, 1\}$, and let f be given by $f(0) = -1$ and $f(1) = 1$. Then f is continuous on A (Exercise 8.7.4), but there is no $x \in A$ where $f(x) = 0$.

2. Let $B = [0, 1] \cup [2, 3]$ and let $f = -1$ on $[0, 1]$ and 1 on $[2, 3]$. Again, f is continuous and changes sign on B, but there is no $x \in B$ where $f(x) = 0$.

Each of these sets has, roughly speaking, a "hole" that allows the function to change sign without ever being 0. In the next two examples we see that the hole doesn't have to be very big.

3. Let $C = (0,1) \cup (1,2)$ and $f(x) = x - 1$. Then f is continuous, negative at $x = 1/2$, and positive at $x = 3/2$, but isn't zero anywhere in C.

4. Let $f : \mathbf{Q} \to \mathbf{R}$ be given by $f(x) = x^2 - 2$. Then f is continuous, it is negative at $x = 0$ and positive at $x = 2$, but f is not zero at any element of its domain.

The last example suggests some of the significance of the Intermediate Value theorem. If we are allowed to think only of rational numbers, we can't even solve a simple equation like $x^2 - 2 = 0$. If we can't solve equations, there is little point in doing mathematics. The Intermediate Value theorem tells us that certain equations have solutions and narrows down where those solutions might be.

If we think of the sets in these examples as bits of string on the number line, there are places where we can take hold of the sets and pull them apart *without having to break anything*. These sets seem to be "disconnected." Hence our definition:

DEFINITION 12.1: A set C is **connected** if every continuous, real-valued function $f : C \to \mathbf{R}$ has the Intermediate Value property on C.

We have seen examples of sets that are not connected. Here are some connected sets:

EXAMPLES 12.1: 5. The Intermediate Value theorem (which we have yet to prove!) says that intervals are connected.

6. A set with a single point is connected. Any function on such a set is continuous and has the Intermediate Value property. (It is impossible for the "values" of the function to have opposite signs.)

The following theorem tells us that examples of connected sets can't be any more complicated than this.

THEOREM 12.2: *If S is a connected subset of the real line, then S is an interval.*

PROOF: Recall that there are nine types of intervals: (a, b), $[a, b]$, $(a, b]$, $[a, b)$, (a, ∞), $[a, \infty)$, $(-\infty, b)$, $(-\infty, b]$, and $(-\infty, \infty)$, where a and b are real numbers. We can select one of these by specifying whether it is

bounded above or below and whether it contains its infimum or supremum. Here we will prove that a *bounded, connected subset of the real line that contains neither its infimum nor its supremum is of the form* (a, b) *for some real numbers* a *and* b. (You are invited to state and prove the other eight parts of the theorem.) Let S be such a set. Since S is bounded, it has an infimum and a supremum, say a and b. We want to show that $S = (a, b)$. This is a set-equality problem, and so we know where we must begin: Let $z \in (a, b)$. Since $a < z < b$, the properties of the infimum and supremum guarantee there are elements x_1 and x_2 of S with $a < x_1 < z < x_2 < b$. Now let $f(x) = x - z$. Then f is continuous, and takes on a negative value at $x_1 \in S$ and a positive value at $x_2 \in S$. Since S is connected, there is a point $t \in S$ with $f(t) = 0$. But $f(x) = 0$ only for $x = z$. Hence $z = t \in S$, and $(a, b) \subseteq S$. By the way a and b were chosen, we know that $S \subseteq [a, b]$. But we are assuming that S contains neither a nor b, meaning that $S \subseteq (a, b)$, and so $S = (a, b)$. ∎

Here we used the Intermediate Value property not to show that some point existed (we already knew z existed), but to say where it was. When you've completed the other eight parts of the proof, you will know that the only connected subsets of the real line are intervals. The Intermediate Value theorem, conversely, says that all intervals are connected, but its proof must wait. Notice that Theorem 12.2 leaves open the possibility that, except for sets with a single point, there are no connected subsets of the real line at all. This may seem very odd, but we will show shortly that the only connected subsets of the *rational* numbers are sets with just one element.

EXERCISES 12.1

1. Complete the proof of Theorem 12.2.

2. Explain why Theorem 12.2 leaves open the possibility that there is no connected set consisting of more than one point.

3. Show that a set S is connected if and only if it has the property that whenever $x < z < y$ and x and y are elements of S, then z is an element of S.

4. (a) If the topological space X has the indiscrete topology, show that every subset of X is connected.

   (b) If X has the discrete topology, show that the only connected subsets of X are those having only one element.

12.2 DISCONNECTIONS

If a set S is *not* connected (we say **disconnected**), there is a continuous, real-valued function on S, say g, that takes on a negative value at some point $a \in S$ and a positive value at some point $b \in S$ but is not 0 in S anywhere between a and b. By making the following adjustment, we may assume that in this case there is a function with domain S that *never* takes on the value 0. Suppose g, a, and b are as above and that $a < b$. Let

$$f(x) = \begin{cases} g(a) & x < a \\ g(x) & a \le x \le b \\ g(b) & x > b. \end{cases}$$

Then f is continuous on the whole real line, fails to have the Intermediate Value property, and never takes on the value 0. Consider the sets $A = f^{-1}((-\infty, 0))$ and $B = f^{-1}((0, \infty))$. Observe that:

(1) A and B are both open since $(-\infty, 0)$ and $(0, \infty)$ are open and f is continuous.

(2) A and B are disjoint since if $x \in A$ then $f(x) < 0$, while if $x \in B$ then $f(x) > 0$, and $f(x)$ can't be both positive and negative.

(3) A and B each contain at least one point of S since $f(a) < 0$ [so $a \in A$] and $f(b) > 0$ [so $b \in B$].

(4) $S \subseteq A \cup B$ since $A \cup B$ contains all points except those where $f(x) = 0$, and we are assuming there are none of these.

We can assemble these observations into a theorem.

THEOREM 12.3: *The set $S \subseteq \mathbf{R}$ is disconnected if and only if there are open sets A and B such that:*

(i) *A and B are disjoint*
(ii) *$A \cap S \neq \emptyset$ and $B \cap S \neq \emptyset$*
and (iii) *$S \subseteq A \cup B$*

If this is the case, we say A and B are a **disconnection** of S. Note that (ii) implies that neither A nor B is empty.

PROOF: We have essentially completed the "only if" part of this proof in the discussion above. Suppose two such sets can be found. Let $a \in A \cap S$ and $b \in B \cap S$. We may assume $a < b$. Since A is an open set, it is a union of open intervals (Theorem 8.11). Suppose $a \in (c, d)$, one of the component intervals of A. Now $d < b$ since $b \notin A$. In our proof

of Theorem 8.11, we saw that $d \notin A \cup B$, and so $d \notin S$. The function $f(x) = x - d$ is continuous on S, $f(a) < 0$ and $f(b) > 0$, but $f(x)$ never takes on the value 0 on S. Thus f fails to have the Intermediate Value property on S. Since such a function exists, S is disconnected. ∎

We have made free use of the Least Upper Bound property in these proofs, but this is acceptable when we consider the position of connectedness in the Big Picture. The idea of disconnectedness will be as useful to us as connectedness. While connectedness is entangled with the order structures of the real line (with its references to intervals and uses of the words "positive," "negative," and "between"), disconnectedness is purely topological. One effect of this will be that many of our proofs concerning connectedness will be done by contradiction or contrapositive. This has often been the case when the negation of a definition gives us better information than the definition itself. While the definition of connectedness tells us something mysterious about continuous functions, Theorem 12.3 gives us something much more tangible (the sets A and B).

EXERCISES 12.2

1. Show that a set is disconnected if and only if it has a proper nonempty subset that is both *open and *closed.

2. Show that the field of formal rational functions is disconnected.

3. Show that the function f, defined in the section by

$$f(x) = \begin{cases} g(a) & x < a \\ g(x) & a \leq x \leq b \\ g(b) & x > b \end{cases}$$

has the properties claimed for it.

12.3 THE BIG THEOREM SAILS INTO THE SUNSET

THEOREM 12.4: *An ordered field with the Least Upper Bound property is connected.*

PROOF: Let f be a continuous function from such a field to the real numbers, and let a and b be such that $a < b$ and $f(a) < 0 < f(b)$. The set $P = f^{-1}((0, \infty))$ is open, as is the set $N = f^{-1}((-\infty, 0))$. Now $N \cap (-\infty, b)$ is nonempty (it contains a) and is bounded above (by b), and so it has a supremum. Let $c = \sup(N \cap (-\infty, b))$. Note that $a \leq c \leq b$.

What is $f(c)$? It can't be positive. Since P is open, if c were in P, there would be an $\varepsilon > 0$ with $y \in P$ for $y \in (c - \varepsilon, c)$, and such a number y would be an upper bound for $N \cap (-\infty, b)$ less than c. In particular, $c \neq b$ since $b \in P$. Likewise, $f(c)$ can't be negative since then c would be in $N \cap (-\infty, b)$, and an open set doesn't contain its supremum. So $c \neq a$. We have shown that $a < c < b$ and that $f(c) = 0$. ∎

COROLLARY 12.5: (The Intermediate Value Theorem) *Intervals in the real line are connected.*

PROOF: With some manipulation, this is essentially the same as the previous proof. If f is a continuous function that fails to have the Intermediate Value property on an interval, one can construct from it a disconnection of all of **R**. ∎

EXERCISES 12.3

1. Complete the proof of Corollary 12.5.

2. (a) Use the Intermediate Value theorem to show that a positive number a has all possible nth roots (that is, the equation $x^n - a = 0$ has a solution). Where does your argument fail if a is negative?

   (b) Show that any polynomial of odd degree has a real root (that is, there is a real number c so that $f(c) = 0$).

3. (a) Let $f : [0, 1] \to [0, 1]$ be continuous. Show that there must be a number x for which $f(x) = x$. (That is, such a function must have a fixed point.)

   (b) Show that this is not true if the function is considered only on the rational numbers.

   (c) Show that the result in (a) is not true if the domain and range of the function are $(0, 1)$ instead of $[0, 1]$.

4. Suppose $f : [a, b] \to \mathbf{R}$ and $g : [a, b] \to \mathbf{R}$ are both continuous and that $f(a) < g(a)$ and $f(b) > g(b)$.

   (a) Draw a picture illustrating this situation.

   (b) Show that there must be a number x such that $f(x) = g(x)$.

   (c) Use this to answer part (a) of the previous exercise.

5. (a) Suppose $f : [0, 2] \to \mathbf{R}$ is continuous and $f(0) = f(2)$. Show that there are elements of $[0, 2]$, say a and b, such that $|a - b| = 1$ and $f(a) = f(b)$.

(b) Suppose $f : C \to \mathbf{R}$ is continuous, where C is the circle $x^2+y^2 = 1$. (We have not defined continuity for such a function precisely, but don't worry about that now.) Show that there are points a and b on C that are diametrically opposed such that $f(a) = f(b)$.[1]

12.4 CLOSING THE LOOP

This time we will "close the loop" (and thereby complete the proof of the Big Theorem) before moving on to other considerations.

THEOREM 12.6: *Any connected ordered field has the Least Upper Bound property.*

PROOF: Suppose the Least Upper Bound property fails. This means there is a nonempty set S that is bounded above but has no supremum. Let $A = \{t : \exists s \in S \ni (t < s)\}$ and $B = \mathbf{R} \backslash A$. We will show that A and B are a disconnection of $\mathbf{R}$. Since S is not empty and is bounded above, neither A nor B is empty. Suppose $x \in A$ and $s \in S$ with $x < s$ (this is how x gets to be in A) and let $\varepsilon = s - x$. If $x - \varepsilon < y < x + \varepsilon$, then $y \in A$. Thus A is open. Now suppose $x \in B$. Then x is an upper bound for S (since no element of S is greater than x), but x is not the least upper bound of S (since S has no least upper bound). Thus there is an $\varepsilon > 0$ such that $x - \varepsilon$ is an upper bound for S. If y is such that $x - \varepsilon < y < x + \varepsilon$, we have $y \in B$, and therefore B is open. Since B is the complement of A, they are disjoint and their union is the whole field. So A and B are a disconnection of the field. ∎

12.5 CONTINUOUS FUNCTIONS AND INTERVALS

Here is another preservation theorem:

THEOREM 12.7: *If S is connected and $f : S \to \mathbf{R}$ is continuous, then $f(S)$ is connected.*

---

[1] An advanced theorem in topology says that if f is a continuous function defined on the surface of a sphere with its outputs in $\mathbf{R}^2$, there are diametrically opposed points where f takes on the same value. In other words, at any give moment, there are two diametrically opposed points on the surface of the earth having the same temperature *and* barometric pressure. In order to interpret the theorem in this way, we must make some assumptions about temperature and pressure.

PROOF: We will use the forward-backward method, a proof by contrapositive, and the characterization of disconnectedness in Theorem 12.3.

▽▽▽

Suppose $f(S)$ is disconnected.

⋆ ⋆ ⋆

Then S is disconnected.

△△△

We apply Theorem 12.3 twice:

▽▽▽

Suppose $f(S)$ is disconnected.

$\hookrightarrow$ There are open sets A and B satisfying Theorem 12.3 relative to $f(S)$.

⋆ ⋆ ⋆

$\hookrightarrow$ There are open sets C and D satisfying Theorem 12.3 relative to S.

Then S is disconnected.

△△△

In the second line, we see *open* sets in the *range* of a continuous function. This immediately suggests our next step:

▽▽▽

Suppose $f(S)$ is disconnected.

There are open sets A and B satisfying Theorem 12.3 relative to $f(S)$.

$\hookrightarrow$ Consider the sets $f^{-1}(A)$ and $f^{-1}(B)$.

⋆ ⋆ ⋆

There are open sets C and D satisfying Theorem 12.3 relative to S.

Then S is disconnected.

△△△

We hope $f^{-1}(A)$ and $f^{-1}(B)$ will play the role of C and D.

▽▽▽

Suppose $f(S)$ is disconnected.

There are open sets A and B satisfying Theorem 12.3 relative to $f(S)$.

Consider the sets $f^{-1}(A)$ and $f^{-1}(B)$.

$\hookrightarrow$ $f^{-1}(A)$ and $f^{-1}(B)$ are open [by definition of continuity].

$\hookrightarrow$ $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint
 [since $\emptyset = f^{-1}(\emptyset) = f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$].

$\hookrightarrow$ $S \cap f^{-1}(A) \neq \emptyset$ and $S \cap f^{-1}(B) \neq \emptyset$
 [since $f(S) \cap A \neq \emptyset$ and $f(S) \cap B \neq \emptyset$].

$\hookrightarrow S \subseteq f^{-1}(f(S)) \subseteq f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

$\hookrightarrow$ Let $C = f^{-1}(A)$ and $D = f^{-1}(B)$.

There are open sets C and D satisfying Theorem 12.3 relative to S.

Then S is disconnected. ∎

△△△

By putting together the two preservation theorems we've proved, we find:

THEOREM 12.8: *If $f : [a, b] \to \mathbf{R}$ is continuous, there exist real numbers c and d so that $f([a, b]) = [c, d]$.*

PROOF: Since $[a, b]$ is compact, $f([a, b])$ is compact. Thus $f([a, b])$ is closed and bounded. Since $[a, b]$ is connected, $f([a, b])$ is connected, and so it is an interval. ∎

Remember Theorem 12.8 as *The continuous image of a closed, bounded interval is a closed, bounded interval*—this has both the Extreme Value theorem and the Intermediate Value theorem as corollaries.

EXERCISES 12.5

1. Suppose $f : [a, b] \to \mathbf{R}$ is a continuous function whose range includes only rational numbers. Show that f is constant.

2. Prove Theorem 12.7 using the definition of connectedness.

3. Explain why the Extreme Value theorem and the Intermediate Value theorem are corollaries to Theorem 12.8.

12.6 A COMMENT ON CALCULUS

All of elementary calculus follows from the Extreme Value theorem and the Intermediate Value theorem, and so in a very real sense Theorem 12.8 embodies the *entire* theoretical content of elementary calculus. If you look through a calculus text for the results that are said to be "beyond the scope of this course," you will find theorems that you can prove (now) using Theorem 12.8. The Intermediate Value theorem is found mainly in results involving approximations to solutions of equations. As we've seen, it is used both to guarantee that solutions to certain equations exist and, in computations like Newton's method, to specify where a solution might be. The consequences of the Extreme Value theorem are more far-reaching:

$$\begin{array}{ccc} & \text{EXTREME VALUE THEOREM} & \\ & \Downarrow & \\ & \text{ROLLE'S THEOREM} & \\ & \Downarrow & \\ & \text{MEAN VALUE THEOREM} & \\ \swarrow\!\!\Downarrow & & \Downarrow \\ \text{TAYLOR'S THEOREM} & & \text{FUNDAMENTAL THEOREM} \end{array}$$

For further contrast between the real and rational numbers, we show:

THEOREM 12.9: *The only connected subsets of the rational numbers are sets with a single element.*

PROOF: We know that sets with one element are connected. Suppose that $S \subseteq \mathbf{Q}$ and that a and b are distinct elements of S. Let z be an irrational number between a and b (by Corollary 6.2.d). Then $A = (-\infty, z)$ and $B = (z, \infty)$ are a disconnection of S. ∎

A set with the property that its only connected subsets are those with one element is called, appropriately enough, **totally disconnected**. Finally, a result promised some time ago:

THEOREM 12.10: *The only subsets of the real line that are both open and closed are the empty set and the whole line.*

PROOF: Suppose A is another set that is both open and closed. Since A is closed, $C(A)$ is open. Since A is neither empty nor the whole line, $C(A)$ is neither empty nor the whole line. Then A and $C(A)$ form a disconnection of the real line, a contradiction. ∎

EXERCISES 12.6

1. Show that the Mean Value theorem fails in the rational numbers.

2. (a) Use the Mean Value theorem to show that any function that is the derivative of another function has the Intermediate Value property (that is, if f is everywhere differentiable, then f' has the Intermediate Value property).

   (b) Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that f is differentiable everywhere (the only place there is any doubt is at $x = 0$).

(c) Show that f' is not continuous at 0 (in (b) you found $f'(0)$, and you can easily find the formula for f' everywhere else).

(d) Conclude that there are discontinuous functions with the Intermediate Value property.

3. Show directly that the connectedness of an ordered field implies it has the Archimedean property. (In this case, "directly" means don't use this argument: *connectedness* $\Rightarrow$ *Least Upper Bound property* $\Rightarrow$ *Archimedean property.*)

4. We have seen that three of the main parts of the Big Theorem (connectedness, the Least Upper Bound property, and the Heine-Borel theorem) imply the Archimedean property. The other three do not. Is there an underlying difference between these two groups of results?

5. In this problem, give two solutions each to (a) and (b). In your first answer, use the fact that connected subsets of the real line are intervals. In your second answer, use either the definition of connectedness or the topological characterization given by Theorem 12.3.

   (a) If S is a connected set, show that S^- (the closure of S) is connected.

   (b) If S is a connected set and T is such that $S \subseteq T \subseteq S^-$, show that T is connected.

   (c) Suppose A and B are connected sets with $A \subseteq B$ and C is such that $A \subseteq C \subseteq B$. Is C necessarily connected?

6. (a) Suppose S is a nonempty open set that isn't the whole real line. Show that there is a cluster point of S that is an element of $C(S)$. Give an example of a set with just one such point.

   (b) Suppose S is a nonempty closed set that isn't the whole real line. Show that S must contain a point of which it is not a neighborhood. Give an example of a set with just one such point. (Go back now and do Exercise 9.6.4 again.)

7. TO CELEBRATE THE COMPLETION OF PART 2: Pick any two parts of the Big Theorem and prove directly that one implies the other (you may have to juggle the Archimedean property). For instance, you might show that "Any ordered field having the Least Upper Bound property is Archimedean and satisfies the Bolzano-Weierstrass theorem." Some of these connections are very difficult, some are easier. There are 30 from which to choose. We've done nine of them, and a couple others have appears as exercises.

Part Three

Topics from Calculus

We now know something about how the real number system works and how it differs from the rational number system. In the next section of the book, we put our knowledge to work in a survey of topics from calculus. These chapters are not intended as an exhaustive review of calculus. We will not discuss the physical applications that make up such an important part of an introductory calculus course. But calculus is, after all, applied real analysis. We approach these topics in such a way as to emphasize what they can tell us about the connections between theory and applications. Applications, in turn, help us to understand the theory better and give us an appreciation of why people might have thought about things in the way they did.

A calculus course usually consists of an intuitive discussion of topics like these (with only occasional proof), along with some historical background (though the latter is often carefully concealed). Here, our main goals are to provide the proofs that are omitted or glossed over in such a course, and examine the *theoretical* background of the subject. You will notice that our approach to these proofs is somewhat different from the one we took in Part 2 of the book. Most calculus proofs are in essence existence questions. As such, they don't lend themselves well to the forward-backward method. We will attack them more directly. Ideally, you should always try to work out these proofs *before* you read the finished versions. Practically speaking, you should study the proofs in the rest of the book by asking why each step is done when it is, by trying to anticipate the steps as much as possible, and by checking to make sure that all the pieces are in their proper places when the proof is claimed to be complete.

Some of the discussions in this part of the book will be familiar, others will not. Some concern techniques more than theory, while others advance the theory in directions we might not have considered before. Since we already know pretty much what calculus is about and have a general idea of how the subject fits together, we don't need to spend much time motivating the study of these ideas.

Chapter 13

Series

13.1 WE BEGIN ON A CAUTIOUS NOTE

Suppose we begin with a sequence, say (a_n), and construct from it another sequence, (s_n), by letting $s_n = a_1 + a_2 + \cdots + a_n$. We considered this process briefly in Exercise 10.2.9, where we saw that the behavior of one sequence like (s_n) can sometimes be deduced from the behavior of another. From this simple idea, we will develop some very powerful tools.

DEFINITION 13.1: A **series** is a sequence constructed in the manner of (s_n) above. The terms of (a_n) are also called the **terms** of the series, while the terms of (s_n) are called the **partial sums** of the series.[1]

THREE NOTES OF CAUTION: 1) It is very important to remember that *a series is a special sort of sequence.* A series and its limit are both denoted $\sum_{n=1}^{\infty} a_n$ or just $\sum a_n$. The former notation will be used when the value of the limit of the series is of some importance, the latter when it is not (which is usually the case in our work). Theorem 13.3 will tell us that, if our only concern is convergence or divergence, we can ignore where the index begins. This notation can be misleading since we refer to a series, before we know whether it converges, with the same symbols we use to denote its limit.

2) The resemblance of a series to an "infinite sum" is a mixed blessing. Series and sums, as we will see, can behave very differently. We will use "sum" to mean "finite sum" and avoid the phrase "infinite sum" as much as possible. On the other hand, we often refer to the limit of a series as its "sum" and write $\sum a_n = S$ instead of $\lim \sum a_n = S$. We also often use the suggestive "S" rather than "L" for this limit. Since this material is familiar from calculus, and since we've already done the difficult part of the work in our study of sequences, this abuse of notation should not

[1] Even if the first index in (a_n) is not 1, we will define s_n to be the sum whose last term is a_n (so the sum defining s_n might not have n terms).

cause serious difficulty.

3) The individual outputs of the function that makes up a sequence are traditionally referred to as "terms." Likewise, numbers being added in a sum are called "terms." But the terms of a series (as defined above) and the terms of the sequence of partial sums (which is what the series really is) are very different things. This double use of the word "term" may be the root of much of the confusion between series and sequences, some of which you undoubtedly felt yourself in your calculus class.

13.2 BASIC CONVERGENCE THEOREMS

We begin our study by restating the main results of Chapters 9 and 10 in the language of series. These are for the most part corollaries to results in those chapters. Those not proved here should be taken as exercises.

DEFINITION 13.2: The series $\sum a_n$, with sequence of partial sums (s_n), **converges to S** if, for every $\varepsilon > 0$, there is an $N_\varepsilon \in \mathbf{N}$ so that $|s_n - S| < \varepsilon$ whenever $n > N_\varepsilon$.

Changing or deleting finitely many terms of a sequence has no effect on its limit, but we must be a bit more careful with series:

THEOREM 13.3: (a) *If k and m are such that all terms in both series are defined, then $\sum_{n=k}^{\infty} a_n$ converges if and only if $\sum_{n=m}^{\infty} a_n$ converges.*
(b) *Changing (or deleting) finitely many terms does not change whether a series converges, but may change its limit.*

PROOF: (a) We may suppose $k > m$. Note that the partial sums of $\sum_{n=k}^{\infty} a_n$ differ from those of $\sum_{n=m}^{\infty} a_n$ by $a_k + \cdots + a_{m-1}$. The result follows from Theorem 9.11.

(b) Suppose $\sum_{n=k}^{\infty} a_n$ and $\sum_{n=j}^{\infty} c_n$ are such that $c_n = a_n$ for $n \geq n_0$. The result is obtained by applying (a) to $\sum_{n=k}^{\infty} a_n$, $\sum_{n=j}^{\infty} c_n$, $\sum_{n=n_0}^{\infty} a_n$, and $\sum_{n=n_0}^{\infty} c_n$. ∎

THEOREM 13.4: *If $\sum_{n=1}^{\infty} a_n$ converges to S, $\sum_{n=1}^{\infty} b_n$ converges to T, and c is a real number, then $\sum_{n=1}^{\infty}(a_n + b_n)$ converges to $S + T$ and $\sum_{n=1}^{\infty} ca_n$ converges to cS.* ∎

More is being said here than meets the eye. If these were finite sums, the first statement would just be the ordinary commutative property, but

the series indicated by $(a_1 + a_2 + \cdots) + (b_1 + b_2 + \cdots)$ can't be obtained from $a_1 + b_1 + a_2 + b_2 + \cdots$ just by rearranging terms.

The question of what happens when we *multiply* series is much more complicated. (It is not even clear how we should go about doing it.) We will deal with multiplication of series in Section 13.11.

EXERCISES 13.2

1. Explain why $(a_1 + a_2 + \cdots) + (b_1 + b_2 + \cdots)$ can't be obtained from $a_1 + b_1 + a_2 + b_2 + \cdots$ just by rearranging terms.

2. Suppose that $a_n > 0$ for all n and let $b_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$. Show that $\sum b_n$ diverges.

3. (a) Show that the insertion of parentheses into a convergent series does not change its convergence or limit. For example, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ might become $(1 + \frac{1}{2}) + \frac{1}{4} + (\frac{1}{8} + \frac{1}{16}) + \cdots$. Let us agree that each set of parentheses can enclose only finitely many terms, and there is no nesting.

   (b) Show that the insertion of parentheses into a divergent series can cause it to converge.

   (c) Suppose that the terms enclosed in any pair of parentheses all have the same sign. Show that resulting series converges if and only if the original series does.

4. Suppose S is an uncountable set of positive real numbers. Show that, for any real number B, there is a finite subset of S whose sum is more than B. (Hint: Think about the cluster points of such a set.)

5. The previous exercise is an important result. It may be rephrased by saying that if an "uncountable" sum converges, it must be the case that all but countably many of its terms are zero. But aren't integrals just convergent "uncountable sums"? Explain.

13.3 SERIES WITH POSITIVE TERMS

Except for a possible different sign in the first term, the sequence (s_n) is increasing if $a_n \geq 0$ for all n and is decreasing if $a_n \leq 0$ for all n. The Monotone Convergence theorem may be restated:

THEOREM 13.5: *Excepting the first term, if $a_n \geq 0$ for all n (or $a_n \leq 0$ for all n), then $\sum a_n$ converges if and only if (s_n) is bounded.* ∎

EXAMPLES 13.3: 1. $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ converges. It is easily seen by an induction that $s_n = \frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n} = 1-\frac{1}{2^n}$, and so $s_n < 1$ for all n. By Corollary 10.4, $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \sup s_n = 1$.

This is an example of a **geometric series**, which are among the few series whose limits we can actually compute. Let $s_n = 1 + r + r^2 + \cdots + r^n$, and so $rs_n = r + r^2 + \cdots + r^{n+1}$. Then $s_n - rs_n = 1 - r^{n+1}$, and $s_n = (1-r^{n+1})/(1-r)$. From this we can see that $\sum_{n=1}^{\infty} r^n$ *converges if and only if* $|r| < 1$, and if so, the limit is $1/(1-r)$. This observation is called the **Geometric Series Test**.

2. $\sum \frac{1}{n}$ diverges. This is called the **Harmonic Series**.[2] Here we will compare the harmonic series with another series that we construct term by term:

$$\begin{array}{ccccccccccccccccccc}
1 & + & \frac{1}{2} & + & \frac{1}{3} & + & \frac{1}{4} & + & \frac{1}{5} & + & \frac{1}{6} & + & \frac{1}{7} & + & \frac{1}{8} & + & \frac{1}{9} & + & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\frac{1}{2} & + & \frac{1}{2} & + & (\frac{1}{4} & + & \frac{1}{4}) & + & (\frac{1}{8} & + & \frac{1}{8} & + & \frac{1}{8} & + & \frac{1}{8}) & + & (\frac{1}{16} & + & \cdots
\end{array}$$

Each term in the harmonic series is at least as large as the one below it. The terms in the bottom series can be gathered into groups whose sum is $1/2$, and so it diverges (its partial sums are unbounded by the Archimedean property). The partial sums of the harmonic series also are unbounded, and so it diverges.

EXERCISES 13.3

1. We saw that $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$. Find a formula for $\sum_{n=k}^{\infty} r^n$, where $k \neq 0$.

2. Give an example of a positive series of *rational* numbers with bounded partial sums that does not have a *rational* limit.

3. (a) Let (a_n) and (b_n) be sequences where the terms of (b_n) are the same as those of (a_n) except that some might be repeated finitely

[2] The numbers $1, \frac{1}{2}, \frac{1}{3}$ …, are referred to as "harmonics," a reference to the role they play in music. If one of two identical vibrating strings is held at a point 1/2 or 1/3 of the way along its length, a chord that "sounds good" is produced. Touching a string at other places simply tends to deaden it. This may account for the fascination of the Greeks with rational numbers, a fascination that gave birth to the subject we now call number theory.

many times. For example, (a_n) might begin 1, 2, 3,..., while (b_n) might begin 1, 1, 2, 2, 2, 3, 3, ... Show that (a_n) converges if and only if (b_n) converges, and if so their limits are the same.

(b) Let $\sum c_n$ and $\sum d_n$ be series in which the terms of $\sum d_n$ are the same as the *nonzero* terms of $\sum c_n$ (for instance, if the first few terms of $\sum c_n$ are $1 + 0 + 2 + 0 + 3 + \cdots$, the first few terms of $\sum d_n$ would be $1 + 2 + 3 + \cdots$). Show that $\sum c_n$ converges if and only if $\sum d_n$ converges, and if so their limits are the same.

(c) Find all places in the chapter (before and after this exercise) where this result is assumed.

13.4 SERIES AND THE CAUCHY CRITERION

If (s_n) is the sequence of partial sums of a series $\sum a_n$ and $n > m$, then $s_n - s_m = a_n + a_{n-1} + \cdots + a_{m+1}$. With a small adjustment to make it easier to read, the Cauchy criterion for series may be stated as follows.

THEOREM 13.6: *The series $\sum a_n$ converges if and only if, for any $\varepsilon > 0$, there is an $N \in \mathbf{N}$ so that $|a_m + a_{m+1} + \cdots + a_n| < \varepsilon$ whenever $n \geq m > N$.* ∎

EXAMPLES 13.4: 1. Theorem 13.6 and the argument of the previous example give us another proof that $\sum 1/n$ diverges. We need only note that

$$\begin{aligned} & \frac{1}{2^{k-1}} + \frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k - 1} + \frac{1}{2^k} \\ > \; & \frac{1}{2^k} + \frac{1}{2^k} + \cdots + \frac{1}{2^k} \text{ (the fraction appears } 2^{k-1} \text{ times)} \\ = \; & 2^{k-1}\left(\frac{1}{2^k}\right) \\ = \; & \frac{1}{2} \end{aligned}$$

For any N, there is a group of terms with indices larger than N that add up to more than $1/2$. The Cauchy criterion fails for any $\varepsilon < 1/2$; the harmonic series diverges.

COROLLARY 13.7: *If $\sum a_n$ converges, then $\lim a_n = 0$.*

PROOF: Apply Theorem 13.6 with $m = n$. ∎

Corollary 13.7 is called the **nth Term Test**, which is usually stated in the contrapositive: *If* $\lim a_n \neq 0$, *then* $\sum a_n$ *diverges.* The harmonic series reminds us that the converse of Corollary 13.7 is not true.

13.5 COMPARISON TESTS

Our goal is usually to decide whether a series converges rather than to find its limit if it does. Among the tools that make this possible is a collection of results referred to as "convergence tests." These fall into two general categories: comparison tests and internal tests. We have already seen two internal tests (the nth Term test and the Geometric Series test) and one comparison test (Exercise 10.2.9, which is usually called simply the **Comparison Test**). We restate the latter here in the terminology of series. We say a series is **positive** if each of its terms is positive.

THEOREM 13.8: *Let $\sum a_n$ and $\sum b_n$ be positive series with $a_n \leq b_n$ for all n. If $\sum b_n$ converges, so does $\sum a_n$; if $\sum a_n$ diverges, so does $\sum b_n$.* ∎

As noted in Exercise 10.2.9, these are the *only* conclusions that can be drawn from these assumptions. In view of Theorem 13.3, the comparison of a_n and b_n need only hold eventually.

EXAMPLES 13.5: 1. $\sum \frac{1}{2^{n^2}}$ converges since $\frac{1}{2^{n^2}} < \frac{1}{2^n}$ for all n and $\sum \frac{1}{2^n}$ converges.

2. $\sum \frac{1}{(\ln(n))^{31}}$ diverges. From calculus, we know that $\lim \frac{\ln(n)}{n^{1/31}} = 0$. This means that, for n sufficiently large, $\frac{\ln(n)}{n^{1/31}} < 1$ and $\frac{1}{(\ln(n))^{31}} > \frac{1}{n}$ (though it might not be immediately obvious just how large is "sufficiently large" in this case).

It is often very difficult to meet the rather demanding hypotheses of the Comparison test (from some point on, *every* pair of terms has to line up properly). A comparison test that could indicate that two series are generally similar might be very useful.

THEOREM 13.9: (The Limit Comparison Test) *Let $\sum a_n$ and $\sum b_n$ be positive series and suppose $\lim(a_n/b_n) = L > 0$. Then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.*

PROOF: It must eventually be true that $0 < (L/2)b_n \leq a_n \leq 2Lb_n$, and

the result follows from the Comparison test, the comment following it, and Theorem 9.11. ∎

You will investigate in Exercise 13.5.1 whether anything can be salvaged from the Limit Comparison test when $L = 0$ or if the sequence of ratios fails to converge.

EXAMPLES 13.5: 3. $\sum \frac{\arctan n}{n}$ diverges. Comparing this to the harmonic series using the Limit Comparison test gives a limit of $\pi/2$ or $2/\pi$. (Since the Limit Comparison test requires only that $L \neq 0$, it doesn't matter which terms are put in the numerator.) We know that the harmonic series diverges, consequently this series diverges.

4. It is difficult to say anything precise about the terms of a series like

$$\sum \frac{14n^3 - 5n^2 + 17n - 2}{6n^4 + 3n^2 - 2n + 11},$$

but we can compare it to the harmonic series using the Limit Comparison test. (Why have we chosen the harmonic series to do this comparison?) The limit would be $14/6$ or $6/14$, and so this series diverges. (The fact that this limit is positive guarantees that the terms in this series are eventually positive.)

EXERCISES 13.5

There are very few exercises in this book that say "Show that this series converges." You should practice using the convergence tests by finding several calculus texts and doing the exercises on convergence in them.

1. Discuss the conclusions, if any, that can be drawn if the limit in the Limit Comparison test is 0 or $+\infty$.

2. Suppose that $\sum a_n$ is a positive series and that $\lim \dfrac{a_n}{1/n^p} = L \neq 0$. That is, the Limit Comparison test has worked for $\sum a_n$ and $\sum 1/n^p$ (but notice that, except for $p = 1$, we don't know yet whether such a series converges or diverges). Show that the Limit Comparison test will *fail* if $\sum a_n$ is compared to any series $\sum 1/n^q$ with $q \neq p$.

3. (a) Let $\sum a_n$ and $\sum b_n$ be positive series with sequences of partial sums (s_n) and (t_n), respectively. If $s_n \leq t_n$ for all n, show that convergence of $\sum b_n$ implies convergence of $\sum a_n$ and that divergence of $\sum a_n$ implies divergence of $\sum b_n$. Explain how this statement differs from the Comparison test.

(b) Show that the Comparison test follows from the result in (a).

(c) Give an example in which this test works but the Comparison test fails.

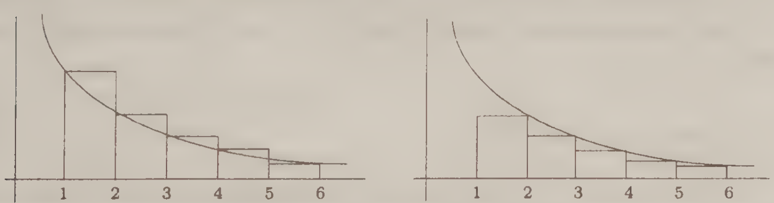
(d) Show that the condition in (a) that the series are positive can't be dropped.

13.6 THE INTEGRAL TEST

Here is a test in which we compare a series with an integral. Recall that we say that the integral $\int_m^\infty f(x)\,dx$ converges if $\lim_{b\to\infty}\int_m^b f(x)\,dx$ exists.

THEOREM 13.10: (The Integral Test) *Suppose that $f(x)$ is a positive, decreasing function whose domain contains some ray $[m,\infty)$. Then the series $\sum_{n=m}^\infty f(n)$ converges if and only if the integral $\int_m^\infty f(x)\,dx$ converges.*

PROOF: This is one of those rare proofs that can be seen all at once by drawing just the right pictures:



The rectangles in the picture on the left have areas $f(1)$, $f(2)$, ... We see that the partial sums of the series are larger than the corresponding "partial integrals" of the integral. If the series converges, its partial sums are bounded above, and so are the partial integrals. The integral converges by Exercise 10.2.8. The rectangles in the second picture are smaller than the corresponding sections of the area under the curve. The areas of these rectangles are $f(2)$, $f(3)$, ... The same sort of argument shows that the convergence of the integral implies that of the sum. ∎

Observe that while the integral test says the series converges if and only if the integral does, they very likely don't converge to the same value.

EXAMPLES 13.6: 1. $\sum 1/n^p$ is called a ***p*-series**. We will examine its behavior for all values of p. We know this series diverges if $p=1$, and if $p\le 0$, the terms of the series do not approach 0. If $p>0$ and $p\ne 1$,

the function $f(x) = 1/x^p$ is positive and decreasing and

$$\lim_{b\to\infty} \int_1^b \frac{1}{x^p}\, dx = \lim_{b\to\infty} \frac{b^{1-p} - 1}{1-p}.$$

The latter limit exists when $p > 1$ but not when $p < 1$. Thus $\sum 1/n^p$ converges if $p > 1$ and diverges if $p \le 1$.

2. The result in the previous example is called the **p-test**. The p-test can be used along with the Limit Comparison test to decide the behavior of series defined by an algebraic function. For instance, the series

$$\sum \frac{\sqrt[3]{4n^2 + 19n + 31}}{\sqrt{9n^7 + 13n^4 + 1}}$$

can be compared using the Limit Comparison test with the series $\sum \frac{1}{n^{17/6}}$. The series $\sum \frac{1}{n^{17/6}}$ converges by the p-test, so this series converges.

3. The series $\sum \frac{1}{n(\ln(n))^p}$ can be seen by the Integral test to converge when $p > 1$ and diverge when $p \le 1$. This illustrates the hazards of using a calculator to examine questions of convergence. If we had $p = 1.0000000000000000001$, it would be very difficult to distinguish $\sum \frac{1}{n(\ln(n))^p}$ from $\sum \frac{1}{n\ln(n)}$ with a calculator (no matter how accurate your calculator might be). Nevertheless, the former series converges and the latter diverges.

EXERCISES 13.6

1. (a) Show that even if the series and the integral in the Integral test both converge, they don't necessarily converge to the same thing.

   (b) Is there *any* circumstance under which the series and the integral would converge to the same thing?

   (c) Examine the pictures used in the proof of the Integral test to find estimates of the value of the integral versus the limit of the series. (Is the integral always greater than the series? Less? By how much?)

2. (a) Show that the hypothesis that $f(x)$ is decreasing can't be omitted from the Integral test.

   (b) The functions in the pictures used to prove the Integral test are both concave up. Is this necessary?

3. Provide the missing details in the proof of Theorem 13.10 (for instance, examine the "partial integrals"). Check that Exercise 10.2.8 has been used correctly.

4. The apparent precision of the p-test might lead us to believe that there is a sharp distinction between the way the terms in a convergent series go to 0 and the way the terms in a divergent series go to 0. In this exercise, we see that this is not the case.

   (a) Let $a_n = 1/n$ and $b_n = 1/(n\ln(n))$. Recall that $\sum a_n$ diverges. Show that $\sum b_n$ diverges and that $\lim(b_n/a_n) = 0$. (Observe that b_n goes to 0 *faster* than a_n, but $\sum b_n$ still diverges.)

   (b) Let $a_n = 1/n^2$ and $b_n = \ln(n)/n^2$. Recall that $\sum a_n$ converges. Show that $\sum b_n$ converges and $\lim(a_n/b_n) = 0$. (In this case b_n goes to 0 *slower* than a_n, but $\sum b_n$ still converges.)

   (c) Suppose $\sum a_n$ is a convergent, positive series. If $B > 0$ and $\varepsilon > 0$ are given, show that there is a natural number N so that $Ba_n + \cdots + Ba_m < \varepsilon$ whenever $n \geq m > N$.

   (d) Suppose $\sum a_n$ is a *divergent*, positive series. If $B > 0$, $N \in \mathbf{N}$, and $\varepsilon > 0$ are given, show that there are natural numbers $n \geq m > N$ so that $\varepsilon a_n + \cdots + \varepsilon a_m > B$.

   (e) Carefully state and prove a theorem to the effect that *For any divergent series, there is a divergent series whose terms go to 0 faster; for any convergent series, there is a convergent series whose terms go to 0 slower.* ("Faster" and "slower" have essentially been defined in this exercise.)

13.7 THE RATIO TEST

The great drawback of comparison tests is that they require us to find a comparable series (or integral) before they can be used. This means in essence that we have to have a good idea what the answer to the question is before we can start. If the behavior of a series escapes our intuition, we may have no indication how to begin. It might be preferable to have tests that can be applied directly to the series in question, without reference to other series. The most familiar of these is the **Ratio Test**.

THEOREM 13.11: *Suppose $\sum a_n$ is a positive series and $\lim(a_{n+1}/a_n)$ exists (say it is equal to L). Then $\sum a_n$ converges if $L < 1$ and diverges if $L > 1$. If $L = 1$, the test gives no information.*

PROOF: We may dispose of the final comment by noting that $L = 1$ for *any* p-series. Some p-series converge, some diverge. Suppose $L > 1$. Note that $(1, \infty)$ is a neighborhood of L, and let N be such that $a_{n+1}/a_n \in (1, \infty)$ for $n > N$. For $n > N$, we have $a_n > a_{n-1} > \cdots > a_N$. Since $a_N > 0$, these terms do not approach 0, and the series diverges by the

nth Term test. Now suppose $L < 1$ and let r be such that $L < r < 1$, and N such that $0 < a_{n+1}/a_n < r$ for $n > N$. Then for $n > N$ we have $a_n < r^{n-N}a_N$ (the pattern can be seen by looking at $a_{N+1}, a_{N+2}, \ldots$), and the proof is completed by using the Comparison test with $\sum a_n$ and the geometric series $\sum(a_N/r^N)r^n$. ∎

The theoretical power of the Ratio test (and of *all* convergence tests) lies in the Comparison test. This should not be surprising since the Comparison test is the bridge between series and the completeness of the real numbers.

EXAMPLES 13.7: 1. $\sum 100^n/n!$ converges since the limit in the Ratio test is 0 (which is less than 1). The Ratio test is especially useful in dealing with series whose terms involve exponentials and factorials.

2. The series $\sum n!/n^n$ converges. The limit in the Ratio test is $1/e < 1$ (a calculation that requires l'Hôpital's Rule).

The Ratio test is essentially a Limit Comparison test with a geometric series whose common ratio r we don't know before the test begins. This is why the ratio test doesn't work on p-series. A p-series doesn't resemble *any* geometric series in the sense of the Limit Comparison test.

EXERCISES 13.7

1. For what, if any, values of p does the series indicated by

$$\left(\frac{1}{2}\right)^p + \left(\frac{1\times 3}{2\times 4}\right)^p + \left(\frac{1\times 3\times 5}{2\times 4\times 6}\right)^p + \cdots$$

converge?

2. (a) Prove the following strengthening of the Ratio test: *Let $\sum a_n$ be a positive series. If* $\limsup_{n\to\infty}(a_{n+1}/a_n) < 1$, *then $\sum a_n$ converges. If* $\liminf_{n\to\infty}(a_{n+1}/a_n) > 1$, *then $\sum a_n$ diverges.*

   (b) Why does this constitute a *strengthening* of the Ratio test?

   (c) State and prove a lim sup-lim inf version of the Limit Comparison test.

13.8 TWO MORE TESTS FOR POSITIVE SERIES

The Ratio test is the best-known internal convergence test, but there are others, each useful in its own way. We will state two. The first can

sometimes be used to decide convergence of a series when the Ratio test has failed. Notice that the limit in this test exists only if $\lim a_{n+1}/a_n = 1$ (that is, if the Ratio test has failed), and that, even so, it must eventually be the case that $a_n/a_{n+1} > 1$, otherwise the terms of the series do not go to 0.

THEOREM 13.12: (Raabe's Test) *Let $\sum a_n$ be a positive series and suppose that*

$$\lim \left[n \left(\frac{a_n}{a_{n+1}} - 1 \right) \right] = L.$$

If $L > 1$, then $\sum a_n$ converges; if $L < 1$, then $\sum a_n$ diverges.

PROOF: We will prove the first statement. The rest is Exercise 13.8.1. Let r be such that $1 < 1 + r < L$ (this choice is made in hindsight to make the proof go more smoothly). Let N be such that $n\left(\frac{a_n}{a_{n+1}} - 1\right) > 1 + r$ when $n \geq N$. Manipulating this inequality, we obtain $na_n - (n+1)a_{n+1} > ra_{n+1}$ for $n \geq N$. Writing out these expressions:

$$\begin{array}{rcl} Na_N - (N+1)a_{N+1} & > & ra_{N+1} \\ (N+1)a_{N+1} - (N+2)a_{N+2} & > & ra_{N+2} \\ & \vdots & \\ Ma_M - (M+1)a_{M+1} & > & ra_{M+1} \end{array}$$

(note that all these expressions are positive). Adding, we obtain

$$Na_N - (M+1)a_{M+1} > r(a_{N+1} + a_{N+2} + \cdots + a_{M+1}).$$

Since $(M+1)a_{M+1} > 0$, we have $Na_N > r(a_{N+1} + a_{N+2} + \cdots + a_{M+1})$. Now r is fixed, and the left side of this inequality does not depend on M. Thus the partial sums of $\sum a_n$ are bounded, and the series converges. ∎

EXAMPLES 13.8: 1. Raabe's test doesn't help us with the harmonic series, but it does correctly decide the behavior of p-series with $p \neq 1$, where the Ratio test fails. For example, if $a_n = 1/n^2$, the expression in Raabe's test simplifies to $(2n+1)/n \to 2$.

Our final convergence test is an internal/comparison test hybrid with a delightfully descriptive name. Like the comparison tests, the convergence of one series is decided by looking at another one, but the second series is generated internally. Its proof is like our first proof of the divergence of the harmonic series, which reminds us that sometimes a technique we create to solve a small problem might be more useful than we realize.

THEOREM 13.13: (The Cauchy Condensation Test) *If $a_n \geq a_{n+1} > 0$ for all n, then $\sum a_n$ converges if and only if $\sum 2^n a_{2^n}$ converges.*

PROOF: We will show the "if" direction; the rest is Exercise 13.8.2. Since (a_n) is decreasing, for any n we have

$$a_{2^n} + a_{2^n+1} + \cdots + a_{2^{n+1}-1} \leq 2^n a_{2^n}.$$

Thus if the partial sums of the series $\sum 2^n a_{2^n}$ are bounded, so are those of $\sum a_n$, and if $\sum 2^n a_{2^n}$ converges, so does $\sum a_n$ ∎

EXAMPLES 13.8: 2. The Cauchy Condensation test provides an easy proof of the p-test. When we convert the series $\sum 1/n^p$ in the manner of the Cauchy Condensation test, we get $\sum 2^n \left(\frac{1}{2^{np}}\right) = \sum \frac{1}{2^{n(p-1)}}$, a geometric series that converges when $p > 1$ and diverges if $p \leq 1$.

EXERCISES 13.8

1. (a) Complete the proof of Theorem 13.12.

   (b) Show that $\sum a_n$ converges if $\liminf \left[n\left(\frac{a_n}{a_{n+1}} - 1\right)\right] > 1$ and diverges if $\limsup \left[n\left(\frac{a_n}{a_{n+1}} - 1\right)\right] < 1$.

2. Complete the proof of Theorem 13.13.

3. (a) Use the Cauchy Condensation test to examine the series $\sum \frac{1}{n\ln(n)}$.

   (b) Use the Cauchy Condensation test to examine $\sum \frac{1}{(n\ln(n))^p}$ for $p \neq 1$.

4. (a) Show that the condition $\lim \left[n\left(\frac{a_n}{a_{n+1}} - 1\right)\right] = L$ (in Raabe's test) implies that (na_n) is eventually decreasing.

   (b) Show that the condition "(na_n) is eventually decreasing" is not sufficient to guarantee convergence of $\sum a_n$.

   (c) Show that "$\lim na_n = 0$" is not sufficient to guarantee convergence of $\sum a_n$.

   (d) Suppose $\varepsilon > 0$. Show that "$\lim(n^{1+\varepsilon})a_n = 0$" *does* guarantee that $\sum a_n$ converges.

   (e) Suppose (ε_n) is a sequence of numbers such that there is an $\varepsilon > 0$ with $\varepsilon_n > \varepsilon$ for all n. Show that the condition "$\lim(n^{1+\varepsilon_n})a_n = 0$" guarantees convergence of $\sum a_n$.

   (f) Suppose (ε_n) is a sequence of positive numbers with $\lim \varepsilon_n = 0$. Does the condition "$\lim(n^{1+\varepsilon_n})a_n = 0$" guarantee convergence of

$\sum a_n$? Are there conditions that could be put on the sequence (ε_n) that would change this result (either way)?

5. Given a sequence (a_n) with $a_n > 0$ for all n. Let $\rho = \limsup \sqrt[n]{a_n}$.

   (a) Show that $\sum a_n$ converges if $\rho < 1$ and diverges if $\rho > 1$.

   (b) Show that no conclusion can be drawn if $\rho = 1$. (This is called the **Root Test**. It will play an important role in Chapter 15.)

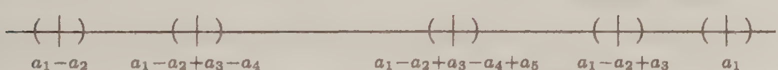
13.9 ALTERNATING SERIES

Our results so far have been applicable mainly to positive series. Now we will look at series having both positive and negative terms. In the simplest such situation, the terms of the series alternate signs. Not only does this make it easy to keep track of the signs, such series can be dealt with by a remarkably simple convergence test.

DEFINITION 13.14: A series is **alternating** if it can be written in the form $\sum(-1)^n a_n$ or in the form $\sum(-1)^{n+1} a_n$, where $a_n > 0$.([3])

THEOREM 13.15: (The Alternating Series Test) *If $a_n \geq a_{n+1} > 0$ for all n and $\lim a_n = 0$, then $\sum(-1)^n a_n$ and $\sum(-1)^{n+1} a_n$ converge.*

PROOF: We will assume the first term in the series is positive. Here is a diagram showing the first few partial sums of such a series:



We see that $s_1 \geq s_3 \geq s_5 \ldots$ and $s_2 \leq s_4 \leq s_6 \ldots$. The intervals $[s_{2n}, s_{2n-1}]$ form a nest of closed, bounded intervals, and the length of $[s_{2n}, s_{2n-1}]$ is a_{2n}, which approaches 0. Let $\{S\} = \bigcap_n [s_{2n}, s_{2n-1}]$ and let $\varepsilon > 0$ be given. If N is chosen so that $a_{2N} < \varepsilon$, we have $[s_{2n}, s_{2n-1}] \subseteq [s_{2N}, s_{2N-1}] \subseteq (S - \varepsilon, S + \varepsilon)$ for $n > N$. It follows that $\sum(-1)^{n+1} a_n = S$. ∎

COROLLARY 13.16: *If $\sum(-1)^n a_n$ or $\sum(-1)^{n+1} a_n$ is an alternating series with limit S, then $|s_n - S| \leq a_n$ for each n.* ∎

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[3] If we say "$\sum a_n$ is an alternating series," then the a_n's we have just written down are *not* the a_n's in this definition. When we are discussing alternating series, we will always assume that the series has been written in the form suggested in the definition, so that $a_n > 0$ for all n.

(Observe that this is a corollary to the *proof* of Theorem 13.15, not to the result itself.)

EXAMPLES 13.9: 1. $\sum \frac{(-1)^n}{n^2}$ converges by the Alternating Series test.

2. $\sum \frac{(-1)^{n+1}}{n}$ converges by the Alternating Series test. This is called the **Alternating Harmonic Series**. We can use Corollary 13.16 to find an approximation to its sum. Adding 200 terms gives us a sum of approximately 0.690, which is within 1/200 of the actual limit. This seems to be a lot of work to do for very little accuracy.

3. $\sum \frac{\sin n}{n^2}$ is *not* an alternating series, and so the test will not work. We can show that it converges anyway, using Theorem 13.6. Note that

$$\left|\frac{\sin n}{n^2} + \cdots + \frac{\sin m}{m^2}\right| \leq \frac{1}{n^2} + \cdots + \frac{1}{m^2},$$

which may be made as small as we wish because $\sum \frac{1}{n^2}$ converges.

EXERCISES 13.9

1. (a) Construct a proof of the Alternating Series test based on the Monotone Convergence theorem (rather than the Nested Intervals property).

   (b) Construct a proof of the Alternating Series test based on the Cauchy criterion.

   (c) Construct a proof of the Alternating Series test based on the Least Upper Bound property.

   (b) Construct a proof of the Alternating Series test based on the Bolzano-Weierstrass theorem.

2. (a) Show that the numbers sin(1), sin(2), sin(3), ..., don't alternate signs.

   (b) Show that no more than four of these in a row can have the same sign.

   (c) Does it ever happen that four in a row do have the same sign?

3. (a) Does the picture drawn in the proof of Theorem 13.15 support the claims made about it? That is, could the situation be different enough from the one shown to change any of the results?

   (b) Draw a picture that describes the Alternating Series test if the first term in the series is negative.

4. Show that the condition $a_n \geq a_{n+1}$ in Theorem 13.15 is necessary. (That is, construct an alternating series whose term go to zero that *diverges*.)

5. (a) Show that the series indicated by $1+1/2-1/3-1/4+1/5+1/6-\cdots$ converges. (This is the harmonic series with signs changing every second term.)

   (b) Suppose we insert the signs $+,+,+,-,-,-,+,+,+,\ldots$, in the harmonic series. Show that the resulting series converges.

   (c) Suppose we insert the signs $+,-,-,+,-,-,\ldots$, in the harmonic series. Show that the resulting series *diverges*.

   (d) Note that in parts (a) and (b), the number of + signs that appear before a given term in the series is always just about the same as the number of − signs. In part (c), though, the number of + signs is always about half the number of − signs. Consider the following (whose proof is beyond the scope of this book): Suppose that for each term of the harmonic series we flip a fair coin, inserting a + before the term if the coin comes up heads, and a − if it comes up tails. In this way, the number of + signs and the number of − signs will be, in the long run, about the same (this is what "fair" means). Does such a series converge?

13.10 ABSOLUTE AND CONDITIONAL CONVERGENCE

In Example 13.9.3, we decided the convergence of a series by observing that the series consisting of the *absolute values* of its terms converges.

DEFINITION 13.17: The series $\sum a_n$ is said to **converge absolutely** (or to be **absolutely convergent**) if the series $\sum |a_n|$ converges.

If a series has any negative terms, whether it is absolutely convergent depends upon the behavior of a *different* series (if a series has only finitely many negative terms, Theorem 13.3 shows that its convergence and that of its "absolute" series are equivalent). The previous example leads us to suspect:

THEOREM 13.18: *If $\sum a_n$ is absolutely convergent, it is convergent.*

PROOF: We need only notice that the argument used in Example 13.9.3 is very general: For $n > m$, we have $|a_n + \cdots + a_m| \leq |a_n| + \cdots + |a_m|$. If the series converges absolutely, the right side can be made as small as

desired by Theorem 13.6. Applying Theorem 13.6 to the left side gives us our result. ∎

Students of calculus often find it hard to appreciate Theorem 13.18, since superficially it doesn't seem like much is being said. It is important to remember that, despite their similar names, *convergence* and *absolute convergence* are very different concepts.

The alternating harmonic series (Example 13.9.2) converges by the Alternating Series test, but we have shown (repeatedly!) that it does not converge absolutely. Such a series is called **conditionally convergent**, for reasons that will be clear shortly. Nothing we've done so far would suggest that finding the limit of a series and simply adding terms are really different. The behavior of conditionally convergent series will show us just how different these processes can be.

DEFINITION 13.19: Suppose $\pi : \mathbf{N} \to \mathbf{N}$ is one-to-one and onto. The series $\sum a_{\pi(n)}$ is called a **rearrangement** of $\sum a_n$.

An obvious adjustment must be made in this definition if the index of the series does not begin with 1. The letter π is used here to remind us of "permutation."

THEOREM 13.20: *If $\sum a_n$ is absolutely convergent and $\sum a_{\pi(n)}$ is a rearrangement of $\sum a_n$, then $\sum a_{\pi(n)}$ also converges absolutely and $\sum a_{\pi(n)} = \sum a_n$.*

PROOF: Let us denote $\sum a_{\pi(n)}$ by $\sum b_n$ and let $A = \sum |a_n|$. The sum of *any* collection of the terms $|a_n|$ is bounded above by A, and so

$$\begin{aligned} & |b_1| + \cdots + |b_n| \\ = \; & |a_{\pi(1)}| + \cdots + |a_{\pi(n)}| \\ \leq \; & A \end{aligned}$$

Thus $\sum b_n$ is absolutely convergent, and so it converges. Let the partial sums of $\sum a_n$ be (s_n) and the partial sums of $\sum b_n$ be (t_n), and let $S = \sum a_n$ and $T = \sum b_n$. For a given $\varepsilon > 0$, choose N_0 and $N \geq N_0$ so that the following all hold:

(i) $|s_n - S| < \varepsilon/4$ for $n > N_0$;
(ii) $|t_n - T| < \varepsilon/4$ for $n > N_0$;
(iii) all the terms of s_{N_0} appear in t_N;
and (iv) all the terms of t_{N_0} appear in s_N.

The first two conditions guarantee that any sum of a_n's or b_n's, all of whose subscripts are larger than N_0, is less than $\varepsilon/4$ in absolute value. Then

$$\begin{aligned} & |S-T| \\ = \quad & |(S-s_N)+s_N-(T-t_N)-t_N| \\ \leq \quad & |S-s_N|+|T-t_N|+|s_N-t_N|. \end{aligned}$$

Now $s_N - t_N$ consists of a_n's and b_n's having subscripts larger than N_0 (all the terms with smaller subscripts cancel). Since $N \geq N_0$, this last expression is not larger than

$$\varepsilon/4+\varepsilon/4+|\sum(\text{leftover } a_n\text{'s})|+|\sum(\text{leftover } b_n\text{'s})|<\varepsilon,$$

and so $S = T$. ∎

The reassuring behavior of absolutely convergent series is in sharp contrast to Theorem 13.22 below. First we need a lemma. Given a series $\sum a_n$ (which we assume to have infinitely many positive and infinitely many negative terms), we denote by $\sum p_n$ the series consisting of only the *positive* terms of $\sum a_n$ and by $\sum q_n$the series consisting of only the *negative* terms of $\sum a_n$.

LEMMA 13.21: (a) *$\sum a_n$ converges absolutely if and only if both $\sum p_n$ and $\sum q_n$ converge (and if this is so, $\sum a_n = \sum p_n + \sum q_n$).*
(b) *If $\sum a_n$ converges conditionally, both $\sum p_n$ and $\sum q_n$ diverge.*

PROOF: (a) Suppose $\sum a_n$ converges absolutely. Let the partial sums of $\sum a_n$ be (s_n) and the partial sums of $\sum p_n$ be (u_n). If k_n is chosen so that all the terms of u_n appear in s_{k_n}, we have $u_n = |u_n| \leq |a_1| + \cdots + |a_{k_n}| \leq \sum |a_n|$, and so $\sum p_n$ converges. That $\sum q_n$ converges is shown similarly. That $\sum a_n = \sum p_n + \sum q_n$ is left as Exercise 13.10.1. Now suppose $\sum p_n$ and $\sum q_n$ both converge. The terms of $\sum q_n$ are all negative, and so it converges absolutely (be sure you see why this is so). Any partial sum of $\sum |a_n|$ is bounded by the sum of appropriately chosen partial sums of $\sum p_n$ and $\sum |q_n|$, all of which are bounded, so $\sum |a_n|$ converges.

(b) There are three possibilities for $\sum p_n$ and $\sum q_n$: both converge, both diverge, or one converges while the other diverges. The first has been shown to force absolute convergence of $\sum a_n$, and so we need only establish that the last forces $\sum a_n$ to diverge. Suppose $\sum p_n$ diverges and $\sum q_n = S$. Any partial sum of $\sum a_n$ is larger than

$$[\text{an appropriate partial sum of } \textstyle\sum p_n] + S.$$

Since the partial sums of $\sum p_n$ are unbounded, so are those of $\sum a_n$, and $\sum a_n$ diverges. ∎

THEOREM 13.22: *If $\sum a_n$ is conditionally convergent, it can be made by a suitable rearrangement to do any of the following*

(a) *converge to any given real number;*

(b) *diverge because its partial sums are unbounded above, below, or both;*

(c) *diverge because its sequence of partial sums has two cluster points (which can be chosen arbitrarily).*

PROOF: We will show (a). This is a proof where too much precision only muddles the issues. You may supply the details if you are doubtful. Let L be a real number (suppose $L > 0$). Since $\sum p_n$ diverges, it has a partial sum larger than L. Take the first such partial sum as the beginning of the rearrangement. Suppose this partial sum adds up to $L + c$. Since $\sum q_n$ diverges (and since its partial sums approach $-\infty$), it has a first partial sum that is less than $-c$. Use these terms as the next part of the rearrangement.

Suppose the sum so far is $L - d$. Deleting the terms we have used from the beginning of $\sum p_n$ does not cause it to converge, and so the remaining series has a partial sum exceeding d. Use these as the next part of the rearrangement, and so on. Observe: We never run out of p's or q's; we never use a p or a q more than once; each time we change from selecting p's to selecting q's we use at least one; and we make this change infinitely often. Because of all of this, our procedure yields a rearrangement of $\sum a_n$. Furthermore, since p_n and q_n both approach 0 (because $\sum a_n$ converges), we "overshoot" L each time by distances that approach 0 (though these overshots may not go monotonically to 0). Thus our rearrangement converges to L. ∎

EXERCISES 13.10

1. Verify the assertions in the proof of Theorem 13.20 that:

   (a) The sum of any collection of the terms $|a_n|$ is bounded by A.

   (b) Any sum of a_n's or b_n's all of whose subscripts are larger than N_0 is less than $\varepsilon/4$ in absolute value.

   (c) $N \geq N_0$.

2. Complete the proof of Lemma 13.21.a.

3. (a) Draw a picture to illustrate the proof of Theorem 13.22.a.

   (b) Mimic the proof of Theorem 13.22.a to find the first 20 terms of a rearrangement of the alternating harmonic series that converges to 2.

4. (a) Complete the proof of Theorem 13.22.

(b) Show that a conditionally convergent series can be rearranged so that its set of sequential cluster points is an arbitrarily chosen finite set.

(c) Show that a conditionally convergent series can be rearranged so that its set of sequential cluster points is $\mathbf{N}$.

(d) Is it possible to rearrange a conditionally convergent series so that its set of sequential cluster points is $\mathbf{Q}$?

(e) Is it possible to rearrange a conditionally convergent series so that its set of sequential cluster points is $\mathbf{R}$?

5. (a) Suppose $\pi : \mathbf{N} \to \mathbf{N}$ is a one-to-one, onto function with the property that there is a natural number N so that $\pi(n) = n$ for $n > N$ (that is, "only finitely many numbers get moved by π"). Show that a rearrangement by π does not change the convergence or limit of any series.

(b) If $\pi : \mathbf{N} \to \mathbf{N}$ is a one-to-one, onto function so that $\pi(n) \neq n$ for infinitely many values of n, is there *necessarily* a conditionally convergent series whose convergence is changed by rearrangement by π?

(c) If your answer to (b) is no, consider whether there is a further condition on a rearrangement π that would guarantee that the convergence of *some* series is changed upon application of π.

(d) If your answer to (b) is yes, consider whether there is a further condition on a rearrangement π that would guarantee that no series has its convergence changed upon application of π.

6. The definition of "subsequence" (Definition 9.17) looks something like the definition of "rearrangement." We might define a "subseries" by saying that $\sum a_{n_k}$ is a **subseries** of $\sum a_n$ if (a_{n_k}) is a subsequence of (a_n).

(a) If the series $\sum a_n$ converges, does any subseries necessarily converge?

(b) Are there conditions on (a_n) that would yield a positive result in (a)?

(c) Is a subseries necessarily a rearrangement? Is a rearrangement a subseries?

(d) Explore possible analogues of Theorem 10.11. When can we draw conclusions about the convergence of a series from the behavior of subseries?

(e) Is the Cauchy Condensation test a statement about subseries?

(f) Remember that a series is a special sort of sequence. Is a subseries necessarily a *subsequence* (of the series)?

13.11 CAUCHY PRODUCTS

In this section we consider multiplication of series. Our goal should be to find a means of defining the product of two series in such a way that, if $\sum a_n = A$ and $\sum b_n = B$, then the product of $\sum a_n$ and $\sum b_n$ is AB. Why are we being so vague about this? Isn't it clear enough that the product of $\sum a_n$ and $\sum b_n$ is $\sum a_n b_n$? We can try this out on series whose limits we know.

We know that $\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$. If we multiply this series by itself in the way we have guessed, we have

$$4 = 2 \times 2 = \left(\sum_{n=0}^{\infty} \frac{1}{2^n}\right) \overset{?}{\times} \left(\sum_{n=0}^{\infty} \frac{1}{2^n}\right) = \sum_{n=0}^{\infty} \frac{1}{4^n} = 4/3,$$

which is certainly not what we want. Upon reflection, though, we see that our guess at a definition of multiplication was silly. Consider series that begin $a_0 + a_1 + \cdots + a_n$ and $b_0 + b_1 + \cdots + b_n$ and the rest of whose terms are 0. The product of such series should be the same as the product of the sums:

$$\begin{aligned} & (a_0 + a_1 + \cdots + a_n)(b_0 + b_1 + \cdots + b_n) \\ = \quad & (a_0 b_0) + (a_0 b_1 + a_1 b_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0) + \cdots + a_n b_n. \end{aligned}$$

Notice the terms at the beginning of this sum. The two subscripts in each term in a group add up to the same number. The subscripts on the a's increase from 0 to this number, while those on the b's decrease from this number to 0. Note also that this sum of subscripts tells us the position of the group in the whole sum. Describing this pattern symbolically gives us:

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_n b_{n-k}\right).$$

This is called the **Cauchy Product** of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$. (We must make adjustments if the first subscripts in the series are not 0.)

EXAMPLES 13.11: 1. Consider the previous example again. The Cauchy product of the series $\sum_{n=0}^{\infty} \frac{1}{2^n}$ with itself is

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{2^k 2^{n-k}}\right)\\
=&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{2^n}\right)\\
=&\sum_{n=0}^{\infty}\frac{n+1}{2^n}.
\end{aligned}$$

The last series is easily seen to converge by the Ratio test. We will use a trick to find its limit. Consider the following array:

$$\begin{array}{ccccccccccc}
1 & + & \frac{1}{2} & + & \frac{1}{4} & + & \frac{1}{8} & + & \frac{1}{16} & + \cdots \\
 & & \frac{1}{2} & + & \frac{1}{4} & + & \frac{1}{8} & + & \frac{1}{16} & + \cdots \\
 & & & & \frac{1}{4} & + & \frac{1}{8} & + & \frac{1}{16} & + \cdots \\
 & & & & & & \vdots & & \vdots &
\end{array}$$

Each row of this array is a geometric series with $r = 1/2$. The limits of the rows are $2, 1, 1/2, 1/4, \ldots$. Adding these, we obtain a geometric series whose limit is 4 (which is, we note, the hoped-for product). On the other hand, if we add the *columns* in this array first, and then add those results, we obtain

$$1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{4}\right)+4\left(\frac{1}{8}\right)+\cdots=\sum_{n=0}^{\infty}\frac{n+1}{2^n}.$$

Thus the limit of the Cauchy product of this series with itself is the product of the limit of the series with itself, as desired. On the other hand ...

2) The series $\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$ converges by the Alternating Series test. The Cauchy product of this series with itself is:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^{k+n-k}}{\sqrt{(k+1)(n-k+1)}}\\
=&\sum_{n=0}^{\infty}(-1)^n\sum_{k=0}^{n}\frac{1}{\sqrt{(k+1)(n-k+1)}}.
\end{aligned}$$

But now

$$
\begin{aligned}
& (k+1)(n-k+1) \\
= \;& n+1+nk-k^2 \\
= \;& \frac{n^2}{4}+n+1-\left(\frac{n^2}{4}-nk+k^2\right) \\
= \;& \left(\frac{n}{2}+1\right)^2-\left(\frac{n}{2}-k\right)^2 \\
\leq \;& \left(\frac{n}{2}+1\right)^2,
\end{aligned}
$$

so that

$$
\begin{aligned}
& \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}} \\
\geq \;& \sum_{k=0}^{n} \frac{1}{n/2+1} \\
= \;& (n+1)\frac{2}{n+2} \\
\to \;& 2 \text{ as } n \to \infty
\end{aligned}
$$

Since $2 \neq 0$, the Cauchy product of $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ with itself diverges by the nth Term test.

Evidently we do not yet know the whole story. In light of the previous section, though, one difference between the series in these two examples should jump out at us: The series $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is *absolutely* convergent, while $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ is *conditionally* convergent.

THEOREM 13.23: *If $\sum a_n$ converges absolutely and $\sum b_n$ converges, then the Cauchy product of $\sum a_n$ and $\sum b_n$ converges. If $\sum a_n = A$ and $\sum b_n = B$, their Cauchy product converges to AB.*

PROOF: This proof is largely a bit of careful bookkeeping. We also see again the standard procedure of splitting a quantity into parts, to be dealt with separately. Let $A_n = \sum_{k=0}^{n} a_k$ and $B_n = \sum_{k=0}^{n} b_k$ and let $R_n = b_n - B$. Then we have

$$\sum_{n=0}^{N} \sum_{k=0}^{n} a_n b_{n-k}$$

$$\begin{aligned}
&= (a_0b_0) + (a_0b_1 + a_1b_0) + \cdots + (a_0b_N + \ldots + a_Nb_0) \\
&= a_0B_N + a_1B_{N-1} + \cdots + a_Nb_0 \\
&= a_0(B + R_N) + a_1(B + R_{N-1}) + \cdots + a_N(B + R_0) \\
&= A_NB + (a_0R_N + a_1R_{N-1} + \cdots + a_NR_0).
\end{aligned}$$

We want to show that $\lim_{N\to\infty} \sum_{n=0}^{N} \sum_{k=0}^{n} a_nb_{n-k} = AB$. Since $\lim A_N = A$, and $B \in \mathbf{R}$ (since $\sum b_n$ converges), we have that $\lim A_NB = AB$. All we need to show is that the expression in parentheses in the last line above goes to 0 as $N \to \infty$.

Let $\varepsilon > 0$ be given. Since $\sum b_n$ converges, we have $\lim R_N = 0$, and so there is a number N_0 so that $|R_N| < \varepsilon$ whenever $N > N_0$. For any such value of N we have:

$$\begin{aligned}
& |a_0R_N + a_1R_{N-1} + \cdots + a_NR_0| \\
\leq\ & |a_0R_N + a_1R_{N-1} + \cdots + a_{N-N_0}R_{N_0}| \\
& \qquad + |a_{N-N_0+1}R_{N_0-1} + \cdots + a_NR_0| \\
\leq\ & \varepsilon(|a_0| + \ldots + |a_{N-N_0}|) + |a_{N-N_0+1}R_{N_0-1} + \cdots + a_NR_0| \\
\leq\ & \varepsilon \sum_{k=0}^{N-N_0} |a_n| + R(|a_{N-N_0+1}| + \cdots + |a_N|),
\end{aligned}$$

where $R = \max\{|R_j| : j = 1, \ldots, N_0 - 1\}$. The sum in the left-hand term is bounded because $\sum a_n$ converges absolutely. Since $\lim a_N = 0$, the expression in parentheses on the right can be made as small as we wish. (Note that the number of terms in the sum does not change as $N \to \infty$.) Thus we may make $|a_0R_N + a_1R_{N-1} + \cdots + a_NR_0|$ small by making N large enough; that is, $\lim_{N\to\infty} |a_0R_N + a_1R_{N-1} + \cdots + a_NR_0| = 0$. ∎

EXERCISES 13.11

1. The trick used to find the limit of the series $\sum_{n=0}^{\infty} \frac{n+1}{2^n}$ can be used to show that the harmonic series diverges. We may write

$$\begin{aligned}
& \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots \\
=\ & \frac{1}{2} + \frac{2}{6} + \frac{3}{12} + \cdots + \frac{n-1}{n(n-1)} + \cdots
\end{aligned}$$

(note that the numerators of the fractions are now 1, 2, 3,. . . , as they were in the other example). Starting here, use the idea in the example to show that the harmonic series diverges.

2. Given a series $\sum x_n$, define a sequence (a_n) by
$$a_n = \frac{nx_1 + (n-1)x_2 + \cdots + 2x_{n-1} + x_n}{n}.$$
(a) Discuss where this formula "came from." (Compare it with the formula in Exercise 9.2.9.)
(b) If $\sum x_n = S$, show that $\lim a_n = S$.
(c) Show that it is possible for $\lim a_n$ to exist when $\sum x_n$ diverges. (This is called **Cesàro summation**. There are more ways to find the limit of a series than just "adding them all up." Compare this further with Exercise 15.5.5.)

3. Consider again the first example following the definition of the Cauchy product. Is there anything in this argument that might require further explanation?

4. Is it *possible* for the Cauchy product of two series to converge to the product of their limits even if neither of them converges absolutely?

5. A **double-ended series** is one of the form $\sum_{n=-\infty}^{\infty} a_n$. Such a series is said to converge if both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=-\infty}^{-1} a_n$ converge. In this case, the limit of the series is the sum of these two values.
(a) Show that the choice of "break point" doesn't matter: The double series converges if and only if both $\sum_{n=-\infty}^{m} a_n$ and $\sum_{n=k}^{\infty} a_n$ converge for any two integers m and k. (But the sum of these two limits is in general the limit of the double-ended series only if $k = m + 1$.)
(b) Show that if $\sum_{n=-\infty}^{\infty} a_n = L$, then $\lim_{k\to\infty} \sum_{n=-k}^{k} a_n = L$.
(c) The result in (b) seems to provide an easier and more intuitive method of checking whether a double-ended series converges, but show that it is possible to have $\lim_{k\to\infty} \sum_{n=-k}^{k} a_n$ exist without having $\sum_{n=-\infty}^{\infty} a_n$ converge.
(d) Show that the results in (b) and (c) are true for $\lim_{k\to\infty} \sum_{n=-k}^{2k} a_n$.
(e) Show that if $f : \mathbf{N} \to \mathbf{N}$ is any strictly increasing function, it is possible to have $\lim_{k\to\infty} \sum_{n=-k}^{f(k)} a_n$ exist without having $\sum_{n=-\infty}^{\infty} a_n$ converge.

6. (a) Show that every series of the form $\sum_{n=1}^{\infty} \frac{d_n}{10^n}$ converges, where d_n can be any digit (0, 1, ..., 9).
(b) Show that any real number can be written $N + \sum_{n=1}^{\infty} \frac{d_n}{10^n}$, with the series as in (a) and N an integer. (These exercises allow us to represent real numbers as decimal expansions using the theory of series rather than nested intervals.)

(c) Repeat (a) and (b) using series having denominators of 2^n (and using digits 0, 1) or 3^n (with digits 0, 1, 2).

(d) Show that any real number can be written $\sum_{n=-\infty}^{\infty} \frac{d_n}{10^n}$, where only finitely many of the d_n's with negative subscripts are nonzero.

(e) Suppose that in part (a), we have $d_n = 9$ for all n. What is the limit?

(f) Consider the possibility of noninteger number bases: Let b be a number larger than 1 but not an integer. Can every real number can be represented in the form $\sum_{n=-\infty}^{\infty} \frac{d_n}{b^n}$? What should the range of choices for the d_n's be? Are there choices of b for which there are real numbers with two such representations? No such representation?

7. Consider the **infinite product** $\prod_{k=1}^{\infty}(1+a_k)$, where we assume that $a_k > 0$. (The reason for the peculiar construction will be evident in a moment.) Such a product is said to converge if the sequence of **partial products**, $p_n = \prod_{k=1}^{n}(1+a_k)$, converges.

   (a) Show that the product converges if and only if $\sum a_n$ converges. (We write the product as we do to make it easy to single out the a_n's.)

   (b) In order for a series to converge, its terms must approach 0, but for an infinite product to converge, its terms must approach 1. Explain.

8. (a) Draw a circle of radius 1. Inscribe in it a square. Inscribe in the square another circle. Inscribe in this circle an octagon. Keep up this way, doubling the number of sides in the polygon at each step. Find the radii of the first few circles. Do the radii approach a positive limit?

   (b) Repeat (a), except instead of doubling the number of sides of the polygon at each step, just increase it by one (triangle, square, pentagon, hexagon, ...).

Chapter 14

Uniform Continuity

14.1 UNIFORM CONTINUITY

Applications of analysis consist in large part of approximation procedures accompanied by estimates of the errors involved in their use.[1] We make the simplest sort of approximation when we use one value of a function as an estimate for another value. Here are two examples:

EXAMPLES 14.1: 1. Let $f(x) = \sin(x)$. By the Mean Value theorem, if x and y are real numbers, there is a real number c between them so that

$$\begin{aligned} & |\sin(x) - \sin(y)| \\ = \;& |f(x) - f(y)| \\ = \;& |f'(c)||x - y| \\ = \;& |\cos(c)||x - y| \\ \leq \;& |x - y|. \end{aligned}$$

If x and y are close together, then $\sin(x)$ and $\sin(y)$ are close together, in a sense made precise by this inequality. The inequality measures the error we suffer when we use the value of $\sin(x)$ as an estimate for the value of $\sin(y)$. Observe that the error depends on the distance between x and y (as we would expect) but *not* on where x and y are on the number line. The same estimate may be used everywhere. For this reason, the estimate is said to be *uniform* (this will be defined formally in a moment). We can't always get a uniform error estimate ...

2. Let $g(x) = x^2$. By the Mean Value theorem again, if x and y are real numbers, there is a real number c between them so that

[1] There are always errors, since we don't need to *approximate* things we can find exactly. A common misconception about applied mathematics is that the work is done when an approximating procedure has been found. But an estimate is useless without a statement about its accuracy.

$$\begin{aligned} & |x^2 - y^2| \\ \leq \quad & |2c||x - y|. \end{aligned}$$

Observe that c (and the right side of the error estimate of which it is a part) must get larger as x and y get larger (since c is between x and y). The error incurred in approximating x^2 by y^2 seems to depend not only on the distance between x and y but on where they are on the number line. It appears that this error estimate is *not* uniform.

The difference between these examples is one of the most subtle issues in calculus. It may not be immediately clear that a uniform estimate is inherently better than one that is not (we will soon see that uniform estimates are crucial in a number of calculus theorems, though). Recall that the function $f : A \to \mathbf{R}$ is continuous everywhere if

$$\forall a \in A \forall \varepsilon > 0 \exists \delta > 0 \ni ((x \in A \text{ and } |x - a| < \delta) \Rightarrow |f(x) - f(a)| < \varepsilon).$$

The existentially quantified variable δ appears after a and ε in the list of variables, and so δ may depend on both of the others. If we don't want δ to change from place to place, we should not allow it to depend on a. We can accomplish this by changing the order of the variables in the list (but don't take the fact that this change is easy to make typographically to be an indication that the concept is not important):

DEFINITION 14.1: The function $f : A \to \mathbf{R}$ is said to be **uniformly continuous** if

$$\forall \varepsilon > 0 \exists \delta > 0 \ni \forall a \in A((x \in A \text{ and } |x - a| < \delta) \Rightarrow |f(x) - f(a)| < \varepsilon).$$

This is probably not something that should be remembered in symbolic form. Rather, we should remember it this way:

A function is uniformly continuous if there is a choice of δ in the definition of continuity that works for every point $a \in A$.

Unfortunately, the topological characterization of uniform continuity is not very useful to us. Uniform continuity is essentially a topic in hard analysis.

The examples at the beginning of the chapter show that $f(x) = \sin(x)$ is uniformly continuous and that $g(x) = x^2$ is not.[2]

---

[2] *For the observant reader*: The first example uses the fact that the derivative of $\sin(x)$ is bounded. You will show in Exercise 14.1.5 that this is sufficient to guarantee uniform continuity, but *not* necessary. We should not be so sure about the second example. There might be a uniform estimate that we have just not found.

EXAMPLES 14.1: 3. Let $h(x) = 1/x$ for $x > 0$. Given $0 < \varepsilon < 1$ and any $\delta > 0$, there is a number n so that $|1/n - 1/(n+1)| < \delta$ but $|h(1/n) - h(1/(n+1))| = |n - (n+1)| = 1 > \varepsilon$. Consequently h is not uniformly continuous.

4. The function $h(x) = 1/x$ *is* uniformly continuous if its domain is taken to be $(1, 2)$. This may be seen as in Example 14.1.1, since $|h'(x)| \leq 1$ on $(1, 2)$. Notice that changing the domain has changed the result. Uniform continuity depends on both the function and the domain.

A uniformly continuous function is also continuous (you will prove this in Exercise 14.1.2). Loosely speaking, this is a statement having the form "More $\Rightarrow$ Less," which should not be surprising. But we know that a continuous function may not be uniformly continuous, so the statement "Continuous $\Rightarrow$ Uniformly Continuous" is false. The latter is of the form "Less $\Rightarrow$ More," which is something we don't find to be true very often. Our goal is to find additional conditions that will allow us to conclude that a continuous function is uniformly continuous. We want to supply the missing part of the statement: "Less and ??? $\Rightarrow$ More." Much mathematical activity is devoted to just this sort of problem. The last two examples indicate that some condition on the domain of the function might be helpful here. The following is the most important theorem on the subject and the only one we will prove.

THEOREM 14.2: *If $f : A \to \mathbf{R}$ is continuous and A is compact, then f is uniformly continuous on A.*

We will examine this proof at some length, not only to prove the result, but as an illustration of how hard analysis is often accomplished. The tools we have available (continuity and compactness) give us specific and limited information. We will make some observations about the structure of the proof and the tools we have and gradually come to see what those tools will do for us and how to make them do it. This is not the forward-backward method per se, but it is good practice in the assembly of a complicated proof. (What follow are some *ideas*. The proof comes later.)

The definition of uniform continuity (the conclusion we seek) is quantified $\forall\varepsilon\exists\delta$. We know that the proof must begin: "Let $\varepsilon > 0$ be given," and end with the selection of a number δ.

The ε we are given (in the definition of uniform continuity) is a measure of the distance between outputs of f. We can use it the same way in the definition of (ordinary) continuity.

By the definition of continuity, there is, for each $a \in A$, a number $\delta(a)$ so that $|f(x) - f(a)| < \varepsilon$ when $x \in A$ and $|x - a| < \delta(a)$. (This is essentially all we get from the definition of continuity.)

To bring compactness into the picture, we need to find an open cover of A. The definition of continuity gives us, for each $a \in A$, an interval, $(a - \delta(a), a + \delta(a))$. These intervals are an open cover of A.

Having found an open cover of A, we should immediately find a finite subcover: There is a finite collection $\{a_1, \ldots, a_n\}$ with $A \subseteq \bigcup_{i=1}^{n}(a_i - \delta(a_i), a_i + \delta(a_i))$.

Considering how we obtained the intervals we've just found, we see that if $x \in (a_i - \delta(a_i), a_i + \delta(a_i)) \cap A$, then $|f(x) - f(a_i)| < \varepsilon$.

The last inequality looks something like what we want. Each $x \in A$ is in one of the intervals $(a_i - \delta(a_i), a_i + \delta(a_i))$, and so one of these inequalities applies to each such x. But the inequality holds only if one of the inputs to the function f is one of the points $a_1, \ldots, a_n$. Uniform continuity doesn't allow such a restriction. Perhaps we can just take two points, say x and y, *both* in the interval $(a_i - \delta(a_i), a_i + \delta(a_i))$. Then

$$\begin{aligned}
& |f(x) - f(y)| \\
= \quad & |f(x) - f(a_i) + f(a_i) - f(y)| \\
\leq \quad & |f(x) - f(a_i)| + |f(a_i) - f(y)| \\
< \quad & \varepsilon + \varepsilon.
\end{aligned}$$

This looks even more like what we want, but there is still a restriction on x and y that isn't of the form $|x - y| < \delta$ (x and y have to be in the same one of these intervals.)

Let us change our tack a bit. The finite collection of intervals we have found gives us a collection of δ's. These are distances measured in the domain of the function. We need just *one* δ. Let's say $\delta = \min\{\delta(a_i)\}$. What can we conclude if we ask only that $|x - y| < \delta$? Unfortunately, we can have $|x - y| < \delta$ without having x and y in the same one of the intervals we found. If x and y aren't together in one of these intervals, there doesn't seem to be much we can say about $|f(x) - f(y)|$.

We need to find a way to force the numbers x and y to be close enough together so that we can be sure they lie in the same interval. We used the distances obtained from the definition of continuity to produce the δ in our last observation (which measures how far apart x and y are). Perhaps we need only pick a smaller value for δ. This is the last idea we will need, and we can assemble the proof.

PROOF: Let $\varepsilon > 0$ be given. For each $a \in A$, let $\delta(a)$ be such that

$$(x \in A \text{ and } |x-a| < \delta(a)) \Rightarrow |f(x) - f(a)| < \varepsilon/2$$

(the $\varepsilon/2$ will take care of the 2ε we found in the third-from-last). The collection of intervals $\{(a - \delta(a)/2, a + \delta(a)/2)\}$ is an open cover of A (dividing by 2 here makes the distances on the x-axis smaller). Since A is compact, there is a finite set $\{a_1, \ldots, a_n\}$ of elements of A, with

$$A \subseteq \bigcup_{i=1}^{n} (a_i - \delta(a_i)/2, a_i + \delta(a_i)/2).$$

Let $\delta = \min\{\delta(a_i)/2\}$. Suppose x and y are such that $|x - y| < \delta$ and let a_i be such that $|y - a_i| < \delta(a_i)/2$ (there is such an a_i for each y, since $\{(a_i - \delta(a_i)/2, a_i + \delta(a_i)/2) : i = 1 \ldots, n\}$ is a cover of A). Then

$$\begin{aligned} & |x - a_i| \\ \leq \quad & |x-y| + |y - a_i| \\ < \quad & \delta + \delta(a_i)/2 \\ \leq \quad & \delta(a_i)/2 + \delta(a_i)/2. \end{aligned}$$

(The idea "if x is close enough to y and y is close enough to a_i, then x is close enough to a_i" was the last piece of the puzzle to fall into place.) Since $|x - a_i| < \delta(a_i)$ and $|y - a_i| < \delta(a_i)$, we have

$$\begin{aligned} & |f(x) - f(y)| \\ \leq \quad & |f(x) - f(a_i)| + |f(a_i) - f(y)| \\ < \quad & \varepsilon/2 + \varepsilon/2 \\ = \quad & \varepsilon. \ \blacksquare \end{aligned}$$

The examples show that a function can be uniformly continuous on a domain that is not compact, and so Theorem 14.2 is not the whole story. Still, "f is continuous on a closed bounded interval" is such a common hypothesis in calculus that this result is still extremely useful.

COROLLARY 14.3: *If $f : [a, b] \to \mathbf{R}$ is continuous, then it is uniformly continuous.* ■

EXERCISES 14.1

1. (a) Show that the function $f(x) = mx + b$ is uniformly continuous for any value of m.

   (b) A straight line whose slope is not 0 is an unbounded function, and so (a) shows that a function need not be bounded to be uniformly continuous. Show, however, that a uniformly continuous function whose domain is a *bounded* set must be bounded.

(c) Show that $f(x) = x^2$ is uniformly continuous if its domain is a bounded set (compact or not).

(d) Show that the conclusion in (c) is true of any polynomial.

2. Show that a uniformly continuous function is continuous.

3. Discuss what happens if we remove the dependence of δ on ε in the definition of continuity. What sort of functions satisfy the statement: $\forall a \in A \exists \delta > 0 \ni \forall \varepsilon > 0((x \in A \text{ and } |x-a| < \delta) \Rightarrow |f(x)-f(a)| < \varepsilon)$?

4. (a) Let $A \subseteq \mathbf{R}$. Show that $f : A \to \mathbf{R}$ is continuous if and only if the following holds: For any neighborhood of 0, say V, and any $a \in A$, there is a neighborhood of 0, say $U_{a,V}$, such that
$$f((a + U_{a,V}) \cap A) \subseteq f(a) + V$$
(remember that $x + S = \{x + s : s \in S\}$).

(b) Let $A \subseteq \mathbf{R}$. Show that $f : A \to \mathbf{R}$ is uniformly continuous if and only if the following holds. For any neighborhood of 0, say V, there is a neighborhood of 0, say U_V, such that $f((a+U_V) \cap A) \subseteq f(a)+V$ for all $a \in A$ [note that U_V can't depend on a]. (This is a topological characterization of uniform continuity. It's not entirely topological, though, since it contains a reference to addition.)

5. (a) If $f : \mathbf{R} \to \mathbf{R}$ is everywhere differentiable and $f'(x)$ is bounded, show that f is uniformly continuous.

(b) Give an example of a uniformly continuous function that is not everywhere differentiable. (Uniform continuity does not imply differentiability.)

(c) Give an example of a uniformly continuous, differentiable function with an unbounded derivative.

6. (a) Show that the function $h(x) = 1/x$ is uniformly continuous on any set A such that $0 \notin A^-$ (A^- is the closure of A—see Exercise 8.3.2). In particular, $h(x)$ is uniformly continuous on $\mathbf{R}\backslash(-h, h)$ for any $h > 0$.

(b) If f is a continuous function, is there necessarily an open interval (or a set of open intervals) whose length is "small" and such that f is uniformly continuous on the complement of their union? (This is fairly easy.)

(c) (This is extremely difficult.) Does the answer to (b) remain the same if the domain of f is assumed to be bounded?

7. (a) Suppose f is uniformly continuous on $A \cup B$. Show that it is uniformly continuous on A and on B.

(b) Suppose A and B are sets, f is uniformly continuous on A and on B, and $A \cap B \neq \emptyset$. Show that f is uniformly continuous on $A \cup B$.

(c) Show that the result in part (a) remains true if the union of two sets is replaced by a union of any collection of sets.

(d) Give an example of a function that is uniformly continuous on each interval $[-n, n]$, $n \in \mathbf{N}$ but *not* uniformly continuous on $\mathbf{R}$. [So the result in (b) is not true for arbitrary unions.]

(e) Give an example of a function that is uniformly continuous on two intervals A and B but not on $A \cup B$. [The condition "$A \cap B \neq \emptyset$" can't be dropped from part (b). Look for a simple answer!]

(f) What extra condition must be placed on the function in part (d) to guarantee that the function is uniformly continuous on $\mathbf{R}$?

8. How can we be sure that the δ in the proof of Theorem 14.2 is positive?

9. Investigate: "If every continuous function whose domain is S is uniformly continuous, then S is compact." (If this is true, "Continuous $\Rightarrow$ Uniformly Continuous" could have been taken to be the definition of compactness. If you find that this *is* true, prove one result from Chapter 11 based on it.)

10. Show that f is continuous at a point a if and only if for any $\varepsilon > 0$ and any number $B > 0$ there is a $\delta > 0$ so that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < B\varepsilon$. Find a place we could have used this.

11. For a function $f : A \to \mathbf{R}$, $a \in A$, and $\varepsilon > 0$ given, let
$$D(\varepsilon, a) = \sup\{\delta : |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon\}$$
(a) Show that f is continuous at a if and only if $D(\varepsilon, a) > 0$ for all $\varepsilon > 0$.

(b) Show that f is uniformly continuous if and only if $\inf_{a \in S} D(\varepsilon, a) > 0$.

12. (a) If $f : \mathbf{R} \to \mathbf{R}$ is uniformly continuous and (x_n) is a Cauchy sequence, show that $(f(x_n))$ is a Cauchy sequence.

(b) Does this remain true if it is only assumed that f is continuous?

13. (a) Give an example of a function whose domain is a bounded interval, where the function is continuous but not uniformly continuous.

(b) Give an example of a function whose domain is a closed interval, where the function is continuous but not uniformly continuous.

14. Is a composition of uniformly continuous functions necessarily uniformly continuous? What about sums? Products? Quotients?

Chapter 15

Sequences and Series of Functions

15.1 POINTWISE CONVERGENCE

To define convergence of a sequence of real numbers, we needed only to have a way to say when two of them were close together. Ideas of closeness are provided by both the topological and the metric structures of the real line (fortunately, if points are close in the metric sense they are also close in the topological sense, and vice versa). To extend the study of sequences to a study of series, we needed only to understand addition. It would seem, then, that we can construct a theory of sequences in any setting in which we have an idea of closeness, and a theory of series in any setting in which we also know how to add. The vector spaces $\mathbf{R}^n$, for example, fit this description (but sequences and series in $\mathbf{R}^n$ have little to add to our knowledge at this time).

Functions are more interesting, though. We certainly know how to add functions, but what does it mean for two of them to be close together? Consider a sequence of functions,[1] say (f_n) (we assume all the functions in a sequence have the same domain). If x is an element of the domain, then $(f_n(x))$ is a sequence of *numbers*, which might converge for some choices of x and diverge for others. This observation leads to the following definition:

DEFINITION 15.1: The sequence (f_n) **converges pointwise** on the set S to the limit function f if $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in S$.

Here we denote the limit process "$\lim_{n\to\infty}$" (we had just said "lim" when studying sequences of constants), because there are two variables present, and we will often want to consider limits involving x as well as n.

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[1] A sequence of functions is actually a function whose domain is the natural numbers and whose range is some set of functions, but we need not deal in this level of technicality now.

EXAMPLES 15.1: 1. Let $f_n(x) = x^n$. We may take the common domain to be $[0,1]$. Then (f_n) converges for all $x \in [0,1]$. If we let

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } \quad x = 1, \end{cases}$$

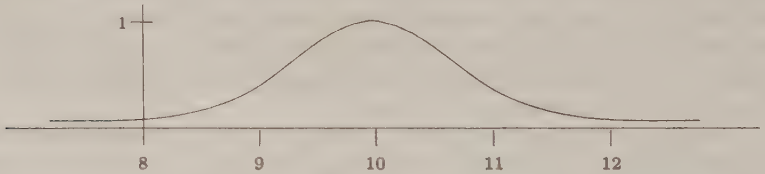
then $f_n \to f$ pointwise on $[0,1]$. Observe that f is discontinuous, even though each of the functions f_n is continuous.

2. Let $f(x)$ be any unbounded function. Let $f_n(x)$ have the same domain as f and be given by

$$f_n(x) = \begin{cases} f(x) & \text{if } |f(x)| \le n \\ n & \text{if } \ f(x) > n \\ -n & \text{if } \ f(x) < -n. \end{cases}$$

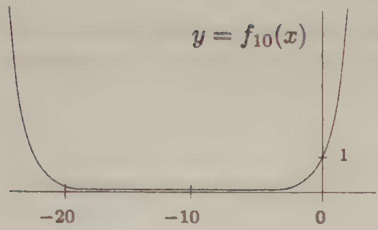
Then $f_n \to f$ pointwise (because of the Archimedean property). Note that each function f_n is bounded. If we take, for example, $f(x) = 1/x$, each f_n is Riemann integrable, but f is not (this kind of construction is often used to define the integral of an unbounded function).

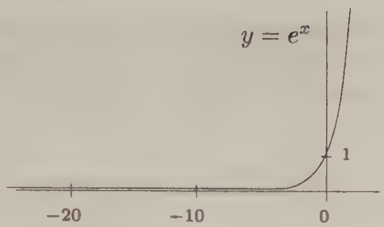
3. Let $f_n(x) = e^{-(x-n)^2}$. Here is part of the graph of $f_{10}(x)$:



As n increases, this "bump" moves to the right. The value of $f_n(x)$ for a particular (large, positive) value of x begins near 0, increases to near 1, and then decreases, approaching 0 as $n \to \infty$. If $x \le 1$, $f_n(x)$ simply decreases to 0. For any $x \in \mathbf{R}$, $\lim_{n\to\infty} f_n(x) = 0$, that is, f_n converges pointwise to the function $f(x) = 0$. Note, however, that for any particular value of n, $f_n(n) = 1$. (Roughly speaking, this means that there are always points on the graphs that are "far away" from where they will be in the limit.)

4. Let $f_n(x) = (1 + x/n)^n$. We know from calculus that $\lim_{n\to\infty} f_n(x) = e^x$. Below are the graphs of $f_{10}(x)$ and e^x drawn to the same scale. Since f_n is a polynomial for each n and $\lim_{n\to\infty} f_n(x) = e^x$ is an exponential function, we have $\lim_{x\to-\infty} \lim_{n\to\infty} f_n(x) = 0$ while $\lim_{x\to-\infty} |f_n(x)| = \infty$ for all n. This is an extreme case of the phenomenon seen in Example 3. In this case there are always points on any of the graph $y = f_n(x)$ that are *extremely* far from their limits.





5. The derivative of the function $f(x)$ may be thought of as the pointwise limit of the sequence $F_n(x) = n[f(x + 1/n) - f(x)]$ (be sure you see why this is so). If we can find any property that must carry over from the functions in a sequence to the limit function, this property must hold for any function that is the derivative of another. Unfortunately, as these examples indicate, such properties are hard to come by. We will discuss some positive results along these lines in Chapter 18.

EXERCISES 15.1

1. Explain how the Archimedean property is used in Example 15.1.2.

2. (a) Explain how the limit of the sequence F_n in Example 15.1.5 represents the derivative of f.

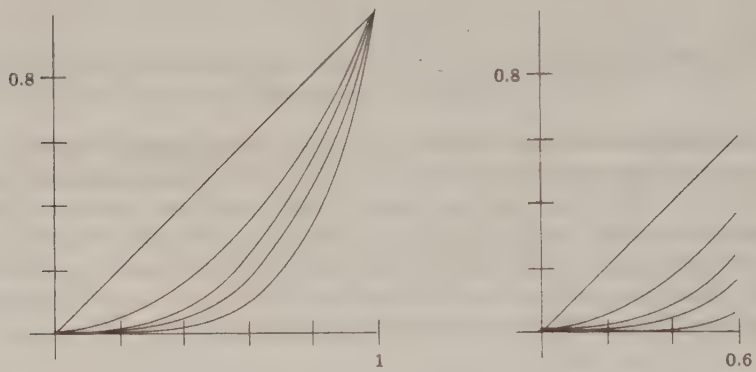
   (b) Draw some examples that illustrate this limit. If you have access to a computer-graphing program, you can use virtually any function for f; if you do not, try $f(x) = x^2$, $f(x) = x^3$, and $f(x) = e^x$

15.2 UNIFORM CONVERGENCE

We have seen that a sequence of continuous functions can converge to a discontinuous functions, a sequence of bounded functions can converge to an unbounded function, and a sequence of integrable functions can converge to a function that is not integrable. In Exercise 15.2.15, you will give an example of a sequence of differentiable functions whose limit is not differentiable. One of the most important topics in the study of sequences of functions is the question of when the properties of the functions in a sequence must carry over to the limit function. Evidence would seem to suggest that the answer is "Not very often." This is our own fault, in part, since we've settled for an easy definition of convergence.

In Definition 15.1 a sequence of functions is taken to be no more than a collection of sequences of numbers. What we really need is an idea of what it means for two *functions* to be close together. Consider the first example again, where we saw that the sequence with $f_n(x) = x^n$

converges pointwise for $x \in [0,1]$ to the function f, where $f(x) = 0$ for $x \in [0,1)$ and $f(1) = 1$. This sequence also converges if we take the domain to be $[0, 0.6]$, but the situations are very different.

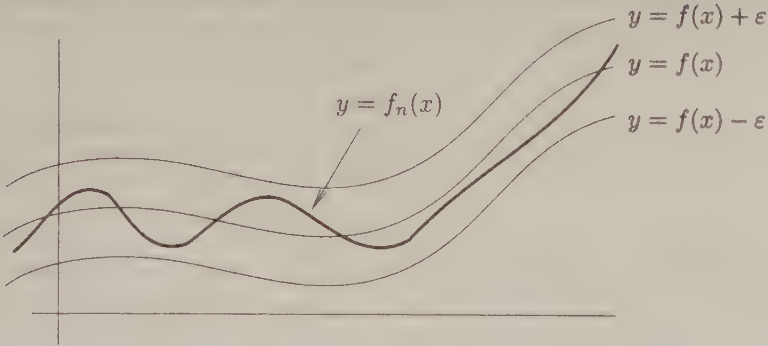


On the left, as in Example 15.1.3, there are values of x (those very close to 1) where, for a given n, the value of $f_n(x)$ is not close to its limit. In fact, for any n, we can find values of $x < 1$ so that x^n is as close to 1 as we like (and so not close to 0). We can do this even though $\lim_{n\to\infty} f_n = 0$ for any such x. In the right hand picture, though, the values of x^n are all closer to 0 than $(0.6)^n$. If we choose n so that $(0.6)^n$ is small, we make *all* the values of x^n close to their limits at the same time. In effect, the *whole function* x^n is close to the limit function 0.

DEFINITION 15.2: The sequence (f_n) **converges uniformly** on the set S to the function f if, given $\varepsilon > 0$, there is a natural number N_ε so that for all $x \in S$, $|f_n(x) - f(x)| < \varepsilon$ whenever $n > N_\varepsilon$.

Note that N_ε can depend on ε (it would be surprising if it didn't) but does *not* depend on x (this is the same sort of change that transformed "continuity" into "uniform continuity"). In the pictures above, convergence is *not* uniform on the left but *is* uniform on the right. We sometimes abbreviate the expression "$f_n(x)$ converges uniformly to $f(x)$" by "$f_n \to f$ uniformly." The condition $|f_n(x) - f(x)| < \varepsilon$ in the definition may be rewritten: $f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon$, and this is something we can draw (see below). The sequence (f_n) converges uniformly to f if the *entire graph* of f_n can be made to lie between the graphs of $f(x) - \varepsilon$ and $f(x) + \varepsilon$ by making n large. As a side benefit, we find that this idea of uniform closeness can be easily translated into a way of measuring the size of a function and the distance between two functions. You will show in Exercise 15.2.9 that the function in Definition 15.3 is really a norm (in

the sense of linear algebra), as its name suggests.



DEFINITION 15.3: The **supremum norm** (or **uniform norm**) of a function $f : S \to \mathbf{R}$ is given by $\|f\|_\infty = \sup_{x \in S} |f(x)|$.

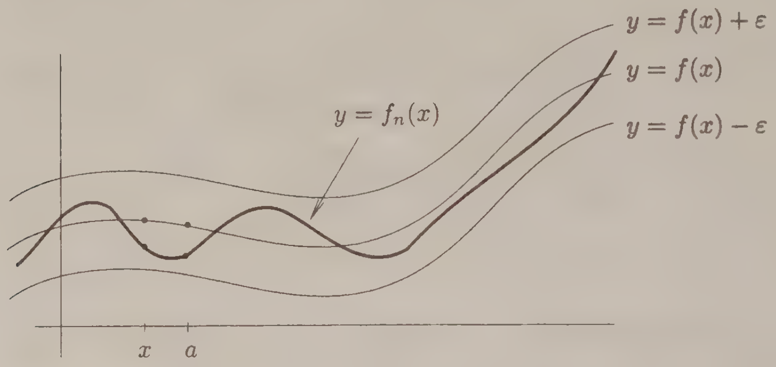
Combining Definitions 15.2 and 15.3, we see that f_n converges to f uniformly if and only if $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$ (hence the name "uniform norm"). Consider again the sequence $f_n(x) = x^n$, and the limit function f defined in the Example 15.1.1. On the interval $[0, 1]$, we have $\|f_n - f\|_\infty = 1$ for all n, while on $[0, 0.6]$ we have $\|f_n - f\|_\infty = (0.6)^n \to 0$. We see again that the convergence is uniform in the latter case, while it is not in the former. (The notation for the supremum norm really should contain some reference to the domain of the function, but we will be able to tell this by context.)

We seem to have found an idea of closeness that considers functions as whole objects. This was certainly our goal, but is the resulting idea of convergence any more useful than pointwise convergence? We have mentioned repeatedly that we might want useful properties of functions in a sequence to carry over to its limit. Often, uniform convergence is just what we need to achieve this.

THEOREM 15.4: *If f_n is continuous on the set S for all n and f_n converges to f uniformly, then f is continuous on S.*

This proof is technical but can be seen easily with the picture below. The four dots near the lower left are all that concern us now. We want to show that the y-coordinates of the top two [which are $f(x)$ and $f(a)$] can be made close together by making x and a close together. But now each of the top two dots can be made close to the one below it because the sequence of functions converges. This can be made to happen simultaneously because the convergence is uniform. Finally, the y-coordinates of

the bottom dots can be made close together because f_n is continuous for all n. Notice in this case how directly these thoughts can be translated into a precise argument.



PROOF: Let $\varepsilon > 0$ be given. Let n_0 be such that $\|f_{n_0} - f\|_\infty < \varepsilon/3$. (This is true for all but finitely many subscripts, but we only need one.) In particular, notice that $|f(x) - f_{n_0}(x)| < \varepsilon/3$ and $|f_{n_0}(a) - f(a)| < \varepsilon/3$. Since f_{n_0} is continuous, there is a $\delta > 0$ so that $|f_{n_0}(x) - f_{n_0}(a)| < \varepsilon/3$ whenever $|x - a| < \delta$. If $|x - a| < \delta$, we have

$$\begin{aligned} & |f(x) - f(a)| \\ \leq \quad & |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(a)| + |f_{n_0}(a) - f(a)| \\ < \quad & \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ = \quad & \varepsilon. \blacksquare \end{aligned}$$

This proof is another example of one of the most important techniques of hard analysis. The quantity that interests us—$|f(x) - f(a)|$ in this case—is seen as a sum of other things, each of which we can estimate. It often happens (as it does here) that the information needed to estimate the different parts of a problem comes from different sources. When a useful estimate has been made of one of the pieces in such an argument, analysts say they have "controlled" that piece (perhaps a bit of wishful thinking).

We can learn much about uniform convergence by thinking about things we *don't* like about the way the functions $f_n(x) = x^n$ approach their limit function f. Here is another example. Let $x_n = 1 - 1/n$. Then $\lim x_n = 1$, and so $f(\lim x_n) = f(1) = 1$. On the other hand, $\lim f_n(x_n) = 1/e \neq 1$. The examination of functions using sequences is very useful (see Theorem 9.16, for example). A situation like this,

where sequences don't seem to do what we want, is inconvenient. Again, uniform convergence comes to the rescue. The proof of this theorem is very similar to the previous one and is left as Exercise 15.2.14. It is good practice in the "divide and conquer" technique.

THEOREM 15.5: *If f_n is continuous on a set S for all n, f_n converges to f uniformly on S, and x_n is a sequence in S with $\lim x_n = x \in S$, then $\lim f_n(x_n) = f(x)$.* ∎

The final theorem in this section is a Cauchy criterion for sequences of functions. Sequences satisfying the hypothesis of this theorem are sometimes called "uniformly Cauchy."[2] This theorem establishes a version of the Cauchy criterion for the set of continuous functions, which means that the set of continuous functions is, in a sense, complete. Since the set of continuous functions is not linearly ordered, it doesn't make sense to talk about the Least Upper Bound property. Nevertheless, we are able to discuss the completeness of the set in terms of another part of the Big Theorem. Having six different characterizations of completeness allows us to pick the one that is the most useful. (Recall that some people take "Cauchy sequences converge" as the definition of completeness.)

THEOREM 15.6: *The sequence (f_n) converges uniformly if and only if, given $\varepsilon > 0$, there is a natural number N_ε so that $\|f_m - f_n\|_\infty < \varepsilon$ whenever $m, n > N_\varepsilon$.*

PROOF: The proof of the "only if" part of this theorem is precisely the same as that of Theorem 10.13, and we won't repeat it. Now suppose that (f_n) is uniformly Cauchy. Since $|f_n(x) - f_m(x)| \le \|f_m - f_n\|_\infty$, for any x, the sequences $(f_n(x))$ are all Cauchy sequences, and therefore each converges. Say $\lim_{n\to\infty} f_n(x) = f(x)$ (we have *defined* f in this way). Now f is the *pointwise* limit of (f_n), and certainly f is the only reasonable candidate for the *uniform* limit of (f_n). We need to show that the convergence of (f_n) to f is uniform. Let $\varepsilon > 0$ be given and let N be such that $\|f_n - f_m\|_\infty < \varepsilon$ whenever $m, n > N$. Then $|f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty < \varepsilon$ for all x. Let n be fixed and greater than N and consider $|f_n(x) - f_m(x)|$ to be a sequence with index m. Letting $m \to \infty$, we have $f_m(x) \to f(x)$, so that $|f_n(x) - f_m(x)| \to |f_n(x) - f(x)| \le \varepsilon$. (There is a $\le$ symbol where we would like to have a $<$, but this detail can be dealt with easily.) Since the last inequality is true for all x, we see that $f_n \to f$ uniformly. ∎

[2] Tradition has saddled us with this unfortunate phrase, but remember that this brilliant mathematician's name is not an adjective.

EXERCISES 15.2

1. Recall that the derivative of a function f may be thought of as the pointwise limit of the sequence $F_n(x) = n[f(x + 1/n) - f(x)]$.

   (a) Define the phrase "uniformly differentiable."

   (b) One of the main objects of the study of sequences of functions is the examination of those properties of the functions in a sequence that are shared by the limit function. Construct and prove a theorem to the effect *If f is uniformly differentiable* (and possibly some other hypotheses), *then f'* (has some nice property).

2. Write the definition of "(f_n) converges pointwise for all x" in symbols and compare it with the definition of uniform convergence.

3. (a) Suppose (f_n) is a sequence of bounded functions and that $f_n \to f$ uniformly. Show that f is bounded.

   (b) Does this remain true if the convergence is not uniform?

4. For each of the following sequences of functions, find the set on which it converges pointwise, and find the limit. Describe a set, if there is one, on which convergence is uniform. (There can be more than one answer to the latter question.)

   (a) $\left(\dfrac{x}{n}\right)$

   (b) $(\sin^n(x))$

   (c) $\left(\dfrac{x^n}{1+x^n}\right)$

   (d) (nxe^{-nx})

   (e) $\left(\dfrac{nx^n}{1+nx^n}\right)$

5. (a) Show that the pointwise limit of a sequence of continuous functions need not have the Intermediate Value property even though each of the functions in the sequence does. (Think of a simple example.)

   (b) Show that the uniform limit of a sequence of continuous functions *does* have the Intermediate Value property. (In view of Theorem 15.4, this is not difficult.)

   (c) Establish the result in (b) without referring to Theorem 15.4.

6. (a) Suppose f is uniformly continuous and let $f_n(x) = f(x + 1/n)$. Show that $f_n \to f$ uniformly.

   (b) Is this true if f is assumed only to be continuous?

7. If f_n is uniformly continuous for all n and $f_n \to f$ uniformly, is f necessarily *uniformly* continuous?

8. (a) Give an example to show that it is possible for a sequence of continuous functions to converge to a continuous function without the convergence being uniform.

   (b) If f is any function at all, show that the function defined by $f_n(x) = f(x)/n$ converges pointwise to the zero function.

   (c) Show that the convergence in (b) is uniform if f is bounded.

   (d) Show that it is possible to have a sequence of *discontinuous* functions converging (pointwise or uniformly) to a continuous function.

9. Show that $|f(x)| \leq \|f\|_\infty$ for all x in the domain of f.

10. Show that the supremum norm is in fact a norm in the sense of linear algebra: If f and g are bounded functions and $k \in \mathbf{R}$, then

    (i) $\|f\|_\infty \geq 0$ for all f, and $\|f\|_\infty = 0$ if and only if $f = 0$
    (ii) $\|kf\|_\infty = |k|\,\|f\|_\infty$
    and (iii) $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

11. Let $f : [a,b] \to \mathbf{R}$ be continuous and define
$$\|f\|_p = \left[\int_a^b (|f(x)|)^p \, dx\right]^{\frac{1}{p}}.$$

    (a) Show that $\|f\|_1$ and $\|f\|_2$ are norms (see the previous exercise).

    (b) Show $\|f\|_p$ is a norm for any $p \geq 1$ (this is quite difficult).

    (c) Show that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$ (hence the notation $\|f\|_\infty$).

12. (a) Show that if $f_n \to f$ uniformly, then $f_n \to f$ pointwise.

    (b) Show that the pointwise limit of a sequence of functions is the only reasonable candidate for the uniform limit, as claimed in the chapter. That is, if $f_n \to f$ pointwise and $g \neq f$, it can't happen that $f_n \to g$ uniformly.

13. Show that $f_n \to f$ uniformly if and only if $\|f_n - f\|_\infty \to 0$.

14. Prove Theorem 15.5.

15. (a) Construct an example of a sequence of differentiable functions that converges uniformly to a nondifferentiable function. (We will see just how dramatically this can happen in Chapter 20.)

    (b) Let $C^1[0,1]$ be the set of functions $f : [0,1] \to \mathbf{R}$ such that f' is continuous. Show that $C^1[0,1]$ is a vector space.

(c) Show that $\|f\|_D = \|f\|_\infty + \|f'\|_\infty$ is a norm on $C^1[0,1]$.

(d) If $f_n \in C^1[0,1]$ for all n and $\|f_n - f\|_D \to 0$, show that f is differentiable and $f_n' \to f'$ pointwise.

(e) Is the convergence you showed in (d) necessarily uniform?

(f) If f_n and f are as in (d), is f necessarily in $C^1[0,1]$? (the only difficult question is whether f' is continuous.)

16. (a) Suppose (f_n) is a sequence of functions on $[0,1]$ that converges to a function f, but that the convergence is not uniform. Show that there *must* be a sequence (x_n) in $[0,1]$ so that $x_n \to x \in [0,1]$ but $f(x_n)$ doesn't converge to $f(x)$.

(b) Show that the sequence (f_n) with $f_n(x) = n^2x^2e^{-nx}$ converges to 0 for all $x \in [0,1]$, but that the convergence is not uniform. Find a sequence as guaranteed in (a).

17. (a) Define the phrases "The series $\sum f_n(x)$ converges" and "The series $\sum f_n(x)$ converges uniformly."

(b) Show that the series $\sum f_n$ converges uniformly if and only if, for any $\varepsilon > 0$, there is an $N \in \mathbf{N}$ so that $\|f_m + \cdots + f_n\|_\infty < \varepsilon$ whenever $n \geq m > N$.

18. (a) If $\sum a_n$ is an absolutely convergent series, show that $\sum a_n \sin(nx)$ converges absolutely and uniformly.

(b) What properties of $\sin(x)$ did you actually use in (a)? If $\sum a_n$ is an absolutely convergent series, what conditions on the functions f_n will guarantee that $\sum a_n f_n(x)$ converges absolutely and uniformly?

19. (a) Consider the set of ordered n-tuples: $\vec{x} = (x_1, x_2, \ldots, x_n)$ with distance given by $d(\vec{x}, \vec{y}) = \max\{|x_i - y_i| : i = 1, \ldots, n\}$. If
$$\left(\vec{x}^{(k)}\right) = \left(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\right)$$
is a sequence of these objects, show that $\vec{x}^{(k)} \to \vec{a} = (a_1, a_2, \ldots, a_n)$ if and only if $x_j^{(k)} \to a_j$ for all j. (Take some time to sort out the notation here. This is why we didn't spend any time talking about sequences of n-tuples.)

(b) Explain why uniform convergence and pointwise convergence are the same for n-tuples.

(c) Repeat part (a) with the distance between n-tuples given by
$$d(\vec{x}, \vec{y}) = \sqrt{\textstyle\sum_{i=1}^n (x_i - y_i)^2}.$$

(d) Show that a sequence of n-tuples converges in the sense given by the distance measurement in (a) if and only if it converges in the sense given by the distance measurement in (c).

(e) Draw some pictures to compare these two concepts of distance as they apply to ordered pairs. Explain (d) in terms of these pictures.

15.3 TOPOLOGY OF FUNCTION SPACES

In Chapter 8 we observed that the topology of a set could be understood by a study of its convergent sequences. Now we have an opportunity to try this out. We have described the convergent sequences in the space of continuous functions but have not discussed what it means for a set of functions to be open. We will examine only one aspect of this large problem. We will ask what it means for a set in the space of continuous functions to be compact.

First we must decide which characterization of compactness will be most useful. To use the definition of compactness would require us to consider functions whose domains are sets of functions. Let's avoid this if we can. To use the Heine-Borel theorem we would have to develop a full theory of open sets, which we've said we don't want to do. Still another characterization of compactness is given in Exercise 11.3.8: A set is compact if every sequence in it has a subsequence that converges to an element of the set.[3] This involves concepts we know about and seems to be just what we need.

We want to avoid discussing the open subsets of the space of functions at any length, but they are not difficult to define. We can measure the size of a function (its supremum norm) and the distance between two of them (the distance between f and g is $\|f-g\|_\infty$). We can then define ε-neighborhoods, and so open sets. We will think about this only long enough to do an example (giving us an idea what we are up against).

The circle of proofs surrounding the Heine-Borel theorem allows us to conclude that a compact set of continuous functions must be closed and bounded. Unlike the situation on the real line, though, this is not enough to guarantee that a set is compact.

EXAMPLES 15.3: 1. Let $\mathcal{B}$ be the set of continuous functions on the interval $[0,1]$ with $\|f\|_\infty \le 1$. This set is bounded in the sense that the supremum norms of the functions in it are all bounded by the same number. We will show that the complement of $\mathcal{B}$ is open. If $f \notin \mathcal{B}$, there is an $x_0 \in [0,1]$ so that $|f(x_0)| > 1$. Let $\varepsilon = (|f(x_0)| - 1)/2$. If g is any continuous function with $\|f-g\|_\infty < \varepsilon$, then $|g(x_0)| > 1$, and so

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[3] This is actually the definition of **sequential compactness**, which is not quite the same as compactness, but the difference need not concern us now. Exercise 11.3.8 says that, on the real line at least, sequential compactness and compactness are the same.

$g \notin \mathcal{B}$. Thus the ε-neighborhood of f, $\{g : \|f - g\|_\infty < \varepsilon\}$, is contained in the complement of $\mathcal{B}$, and so $\mathcal{B}$ is closed. Consider the sequence (x^n), which is contained in $\mathcal{B}$. The limit of this sequence is not continuous, hence is not an element of $\mathcal{B}$, and any subsequence has the same limit. No subsequence converges to an element of $\mathcal{B}$, and $\mathcal{B}$ can't be compact (even though it is closed and bounded).

Evidently, being closed and bounded is not sufficient to guarantee that a set of functions is compact. Another condition must be met, which is defined below. This condition is very subtle, and it is easy to see how it might have escaped our attention.

DEFINITION 15.7: A collection of functions is **equicontinuous** on the set S if, for each $\varepsilon > 0$, there is a $\delta > 0$ so that $|f(x) - f(y)| < \varepsilon$ whenever f is in the collection, x and y are in S, and $|x - y| < \delta$.

Every function in an equicontinuous collection is uniformly continuous. This uniformity extends through the collection in the sense that the δ to be found doesn't depend on either the choice of x and y *or* the choice of the function f (the collection is *uniformly uniformly continuous*!). Remember that on the real line, the union of the range of a convergent sequence and its limit is a compact set. The familiar ring of the following theorem indicates that equicontinuity puts us on the right track.

THEOREM 15.8: *If (f_n) is a uniformly convergent sequence of continuous functions defined on a compact set with $f_n \to f$, then $\{f_n\} \cup \{f\}$ is equicontinuous.*

PROOF: We will first show there is a number N so that $\{f_n : n > N\}$ is equicontinuous.[4] Let $\varepsilon > 0$ be given. Since (f_n) is uniformly convergent, we can find N so that $\|f_n - f\|_\infty < \varepsilon/3$ whenever $n > N$. Note that f is uniformly continuous and let $\delta > 0$ be such that $|f(x) - f(y)| < \varepsilon/3$ whenever $|x - y| < \delta$. Mimicking the proof of Theorem 15.4, we see that $|f_n(x) - f_n(y)| < \varepsilon$. This inequality holds for all x and y with $|x - y| < \delta$ and, more importantly, for all $n > N$. Thus $\{f_n : n > N\}$ is equicontinuous. You will show in Exercise 15.3.4 that (i) A finite set of uniformly continuous functions is equicontinuous (so $\{f_n : n \leq N\}$ is equicontinuous) and (ii) The union of two equicontinuous sets is equicontinuous. Therefore $\{f_n\} \cup \{f\} = \{f_n : n \leq N\} \cup \{f_n : n > N\} \cup \{f\}$ is equicontinuous. ∎

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[4] This proof is very much like those of Theorems 9.7 and 15.4, and it would be a good idea to review those proofs before reading this one.

Our goal is to obtain a theorem that says that a set of functions that is closed, bounded, and equicontinuous is (sequentially) compact. We will need the following lemma.

LEMMA 15.9: *Suppose $S \subseteq \mathbf{R}$ is compact and T is a dense subset of S. If $\{f_n\}$ is equicontinuous on S and $f_n \to f$ uniformly on T, then $f_n \to f$ uniformly on S.*

PROOF: We will show that (f_n) is uniformly Cauchy on S. Let $\varepsilon > 0$ be given and $\delta > 0$ be such that $|f(x) - f(y)| < \varepsilon/3$ for all $f \in S$ whenever $|x - y| < \delta$ (from the definition of equicontinuity). Since S is compact and is the closure of T, there are elements $t_1, \ldots, t_n$ of T so that every element of S is within a distance δ of one of these t's. Since (f_n) converges uniformly on T, there is an N so that $|f_m(t_k) - f_n(t_k)| < \varepsilon/3$ for all $m, n > N$ and all t_k. Let $x \in S$ and let k be such that $|x - t_k| < \delta$. For all n, $|f_n(x) - f_n(t_k)| < \varepsilon/3$ (by equicontinuity). Then, for $m, n > N$,

$$\begin{aligned}
& |f_m(x) - f_n(x)| \\
\leq\ & |f_m(x) - f_m(t_k)| + |f_m(t_k) - f_n(t_k)| + |f_n(t_k) - f_n(x)| \\
<\ & \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\
=\ & \varepsilon.
\end{aligned}$$

Since this is true for all $x \in S$, we have $\|f_m - f_n\|_\infty < \varepsilon$ for $m, n > N$, and so, by Theorem 15.6, (f_n) converges uniformly on S. ∎

Now we can establish our main theorem. The proof of this theorem uses a technique known as "diagonalization" (or "Cantor diagonalization," owing to its pictorial resemblance to the proof of the uncountability of the real numbers). This is such a familiar and useful technique that authors of research publications tend to say "the proof is completed by a diagonalization argument." To be precise, we will say that a set of functions is **closed** if it satisfies Theorem 9.15 and is **bounded** if there is a number B so that $\|f\|_\infty \leq B$ for all f in the set.

THEOREM 15.10: (The Arzelà-Ascoli Theorem) *Let $\mathcal{F}$ be a collection of functions defined on a compact set S. If $\mathcal{F}$ is closed, bounded, and equicontinuous, then every sequence in $\mathcal{F}$ has a subsequence that converges to an element of $\mathcal{F}$.*

PROOF: Let (f_n) be a sequence in $\mathcal{F}$ and let $T = \{t_1, t_2, \ldots\}$ be a countable subset of S whose closure is S. Now $\{f_n\} \subseteq \mathcal{F}$, and so $\{f_n\}$ is bounded (that is, $\{\|f_n\|_\infty\}$ is bounded). Thus $(f_n(t_1))$ is a

bounded sequence of real numbers, and so has a convergent subsequence by the Bolzano-Weierstrass theorem for sequences. We denote this subsequence $(f_{n,1}(t_1))$. Similarly, $(f_{n,1}(t_2))$ is bounded (watch the subscripts closely), and so it has a convergent subsequence, say $(f_{n,2}(t_2))$. Note that $(f_{n,2}(t_1))$ also converges since $(f_{n,2}(t_1))$ is a subsequence of $(f_{n,1}(t_1))$. We continue in this way, to produce a collection of sequences, each of which is a subsequence of the previous one and each converging at all the points that gave rise to those before. We may keep track of these sequences in a table:

$$\begin{array}{ccccc} f_{1,1}(t_1) & f_{2,1}(t_1) & f_{3,1}(t_1) & f_{4,1}(t_1) & \cdots \\ f_{1,2}(t_2) & f_{2,2}(t_2) & f_{3,2}(t_2) & f_{4,2}(t_2) & \cdots \\ f_{1,3}(t_3) & f_{2,3}(t_3) & f_{3,3}(t_3) & f_{4,3}(t_3) & \cdots \\ \vdots & & & & \end{array}$$

Consider the diagonal sequence $(f_{n,n})$. This is eventually a subsequence of $(f_{n,k})$ for any k, and so $(f_{n,n}(t_k))$ converges for all k. Call the limit f. We show that the convergence of $f_{n,n}$ to f is uniform on T. Suppose it is not uniform, then there is a sequence in T, say (τ_n), so that τ_n converges to $\tau \in T$ but $f_{n,n}(\tau_n)$ does not converge to $f(\tau)$. Then there is an $\varepsilon > 0$ so that $|f_{n,n}(\tau_n) - f(\tau)| \geq 2\varepsilon$ for infinitely many values of n. Since $\tau \in T$, $f_{n,n}(\tau) \to f(\tau)$, and for large enough n, $|f_{n,n}(\tau) - f(\tau)| < \varepsilon$. Putting these together, we see that there are arbitrarily large values of n (making τ_n arbitrarily close to τ) for which $|f_{n,n}(\tau_n) - f_{n,n}(\tau)| \geq \varepsilon$, contradicting the equicontinuity of $\mathcal{F}$. Thus $(f_{n,n})$ converges uniformly on T. By Lemma 15.9, $(f_{n,n})$ converges uniformly on S. Since $\mathcal{F}$ is closed, the limit is an element of $\mathcal{F}$. ∎

EXERCISES 15.3

1. Show that the collection $\{\cos nx : n \in \mathbf{N}\}$ is not equicontinuous.

2. Suppose $\{f_\alpha\}$ is a collection of functions defined on a closed interval I such that there is a number B with $|f_\alpha(x)| < B$ and $|f'_\alpha(x)| < B$ for all $x \in I$. Show that $\{f_\alpha\}$ is equicontinuous.

3. (a) Verify that it is possible to select the subset T necessary for the proof of the Arzelà-Ascoli theorem. (Hint: Make a cover of D consisting of intervals of length $1/2$; reduce to a finite subcover; pick a point of T in each of these intervals; now make a cover of intervals of length $1/3, \ldots$)

(b) Consider the role played by the set T in the proof of the Arzelà-Ascoli theorem. Can the result be obtained while putting conditions on this set that are less demanding?

4. (a) If f is uniformly continuous, show that $\{f\}$ is equicontinuous.

   (b) If F and G are equicontinuous, show that $F \cup G$ is equicontinuous.

   (c) Show that a finite set of uniformly continuous functions is equicontinuous.

5. (a) Does every family of continuous functions that is not equicontinuous contain a sequence that converges to a discontinuous function?

   (b) Does the converse of the Arzelà-Ascoli theorem hold? That is, if every sequence in a set of functions has a uniformly convergent subsequence, is the set necessarily equicontinuous?

6. Show that every set of functions that has the covering property is closed and bounded.

7. (Here is an open-ended project.) In determining the distance between two functions in the supremum norm, we look only at the differences between points on the two graphs with the same x-coordinate. But this does not measure the distance from a point on one of the graphs to the other graph. For instance, the difference between y-coordinates of points on the graphs of $f(x) = x$ and $g(x) = x + 1$ is always 1 (and so $\|f - g\|_\infty = 1$), while the distance from any point $(x, f(x))$ to the graph $y = g(x)$ is only $1/\sqrt{2}$. We might measure the distance between two graphs by considering the distances between points on one graph and the other graph. Suppose f and g both have domain D. The distance from a point $(x, f(x))$ to the graph of g would be given by

$$\inf_{t \in D} \sqrt{(x-t)^2 + (f(x) - g(t))^2}.$$

   We can define the "Distance Between" the graphs of f and g to be

$$DB(f, g) = \sup_{x \in D} \inf_{t \in D} \sqrt{(x-t)^2 + (f(x) - g(t))^2}$$

   (a) Compute $DB(f, g)$ for some specific examples. Note that it is possible to have $\|f - g\|_\infty$ very large while $DB(f, g)$ is very small (if f and g are parallel lines with very large slopes, for instance).

   (b) Should there be restrictions on the domain D to help this make more sense? On the functions f and g?

   (c) Is $\|f\|_B = DB(f, 0)$ a norm in the sense of linear algebra? If so, is it the case that $\|f - g\|_B = DB(f, g)$?

(d) If a sequence of functions "converges DB" (so $DB(f_n, f) \to 0$), does it necessarily converge pointwise? Uniformly?

(e) Are the converses of the results considered in (d) true?

(f) Consider "big questions" of convergence such as those posed in this chapter (and later in Chapter 18). For instance, if a sequence of continuous functions converges DB, is the limit function necessarily continuous?

(g) $DB(f, g)$ is certainly harder to compute than $\|f-g\|_\infty$. Assuming satisfactory answers to these questions, is there any good reason for using this idea of the distance between functions as opposed to the uniform norm?

15.4 THE WEIERSTRASS M-TEST

A **series** of functions, as we might guess, is a special sort of sequence (it is the sequence of partial sums formed from another sequence). A collection of theorems on convergence of series of functions can be assembled by combining the results from the first few sections of this chapter with those in Chapter 13. Though the results so obtained are interesting and important, they don't have much new to teach us about the real numbers. Here we will prove only one theorem, which is important in that it connects uniform convergence of series of functions (that is, convergence "as functions") with convergence of series of numbers. The theorem has a bland but traditional name.

THEOREM 15.11: (The Weierstrass M-Test) *Suppose (f_n) is a sequence of functions with common domain D and that (M_n) is a sequence of numbers with $\|f_n\|_\infty \leq M_n$ for all n. If $\sum M_n$ converges, then $\sum f_n$ converges uniformly.*

PROOF: The proof uses the adaptation of Theorem 13.6 that you proved in Exercise 15.2.17. Under the hypotheses, we have, for any m and n,

$$\|f_m + \cdots + f_n\|_\infty \leq \|f_m\|_\infty + \cdots + \|f_n\|_\infty \leq M_m + \cdots + M_n.$$

By Theorem 13.6, the sum on the right can be made as small as we wish by choosing m and n large enough. The Cauchy criterion for series of functions holds, and $\sum f_n$ converges uniformly. ∎

The proof of the following corollary is simple and will be omitted. Notice the resemblance of this result to Theorem 13.18. In this context, though, its significance is more apparent since the series of norms (the "absolute values") is very different from the original series of functions.

COROLLARY 15.12: *If the series $\sum \|f_n\|_\infty$ converges, then the series $\sum f_n$ converges uniformly.* ∎

EXERCISES 15.4

1. Use the Weierstrass M-test to show that:
   (a) $\displaystyle\sum \frac{\sin^n x}{n}$ converges uniformly on $(-1,1)$.
   (b) $\sum (x \ln(x))^n$ converges uniformly on $(1/2, 1]$.
   (c) $\sum ne^{-nx}$ converges uniformly on $[a, \infty)$ for any $a > 0$.
2. (a) Does the series in Exercise 1.a converge uniformly on $[-\pi/2, \pi/2]$?
   (b) Does the series in Exercise 1.b converge uniformly on $(0, 1]$?
   (c) Does the series in Exercise 1.c converge uniformly on all of $\mathbf{R}$?

15.5 POWER SERIES

DEFINITION 15.13: A series of the form $\sum a_n(x-a)^n$ is called a **power series about a**.

Power series and their uses are familiar from calculus. We can study the exponential function, for example, by observing that term-by-term differentiation of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ yields precisely the same series. This is a much deeper statement than it might seem. We will concentrate on those parts of the theory that justify statements like this (but we will not finish examining the issue until Chapter 18).

One key to the study of power series is the analysis of the sets on which they converge. That this question has a complete answer comes as a pleasant surprise. The limit superior was defined in Exercise 10.4.10.

THEOREM 15.14: *Let $\rho = \limsup(|a_n|^{1/n})$ and $R = 1/\rho$ (if $\rho = 0$, let $R = \infty$; if $\rho = \infty$, let $R = 0$). Then $\sum a_n(x-a)^n$ converges absolutely if $|x-a| < R$ and diverges if $|x-a| > R$.* The number R is called the **radius of convergence** of the $\sum a_n(x-a)^n$.

PROOF: Simply note that

$$\begin{aligned} & \limsup \sqrt[n]{|a_n(x-a)^n|} \\ = \; & |x-a| \limsup \sqrt[n]{|a_n|} \\ = \; & |x-a|/R, \end{aligned}$$

and the result follows from the root test (Exercise 13.8.5). ∎

EXAMPLES 15.5: 1. Consider $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$. Here $\rho = \limsup(1/2^n)^{1/n} = \limsup 1/2 = 1/2$. The radius of convergence of the series is 2.

2. You may have observed two things about the series in Example 1: The lim sup construction is unnecessarily complicated since $\lim(1/2^n)^{1/n}$ exists. More importantly, you probably remember series like this from calculus, where you dealt with them a little differently.

3. Here is an example where the full power of Theorem 15.14 is needed: $\sum_{n=0}^{\infty} \cos(n)x^n$. Notice that $(|\cos(n)|)^{1/n} \le 1$ for all n, but for any $\varepsilon > 0$, there are infinitely many values of n so that $|\cos(n)| > 1 - \varepsilon$ (you will show this in Exercise 15.5.1). For each such value of n, $(|\cos(n)|)^{1/n}$ is also very close to 1. Putting these two observations together, we see that, for any $\varepsilon > 0$, there are infinitely many n for which $|\cos(n)|^{1/n} > 1 - \varepsilon$, so that $\limsup(|\cos(n)|^{1/n}) = 1$. Thus $R = 1$. This would have been very difficult to establish by any other method, and it still doesn't tell us anything about the limit of the series.

4. Of course, we do have other methods that are familiar from calculus for dealing with power series. By the ratio test, we see that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely whenever $\lim_{n\to\infty} \frac{|x|^{n+1}/(n+1)!}{|x|^n/n!} < 1$. But this limit is 0 for any value of x, so this series converges for any value of x (that is, $R = \infty$), a possibility allowed by Theorem 15.14. Incidentally, we can conclude from this (and Exercise 10.4.10.g) that $\lim(1/n!)^{1/n} = 0$, a fact that is a bit tricky to show directly.

5. Suppose we construct a series $\sum_{n=0}^{\infty} a_n x^n$ by choosing $a_n = \pm 1$ at random (by flipping a coin, say). Then $|a_n| = 1$ for all n, and so $\rho = 1$ and $R = 1$. We can find the radius of convergence of this series without being able to say anything about its value for any specific x (compare this with Exercise 13.9.5).

Theorem 15.14 narrows down the search for the convergence set of a series considerably. When discussing series of functions, though, convergence (even absolute convergence) is usually not as important as uniform convergence.

THEOREM 15.15: *Suppose the series $\sum a_n(x-a)^n$ has radius of convergence R. If $0 < r < R$ (if $R = \infty$, this is true for any $r > 0$), then $\sum a_n(x-a)^n$ converges uniformly on $[a-r, a+r]$.*

PROOF: Note that $\sum a_n r^n$ converges absolutely for $|x| < R$. If x is in the interval $[a-r, a+r]$, then $|x-a| \le r$. If $n > m$, we have

$$\begin{aligned} & \|a_n(x-a)^n + \cdots + a_m(x-a)^m\|_\infty \\ \le \quad & \|a_n(x-a)^n\| + \cdots + \|a_m(x-a)^m\|_\infty \\ \le \quad & |a_n r^n| + \cdots + |a_m r^m|, \end{aligned}$$

which can be made as small as we like because $\sum a_n r_n$ converges absolutely. The result follows from Exercise 15.2.17. ∎

Every $x \in (a-R, a+R)$ is in an interval $[a-r, a+r]$ for some $r < R$. This means that any such x is in some set where $\sum a_n(x-a)^n$ converges uniformly. Some of the consequences of this will be examined in Chapter 18.

EXERCISES 15.5

1. Show that, for any $\varepsilon > 0$, there are infinitely many values of n so that $\cos(n) > 1 - \varepsilon$.

2. Show that the radius of convergence of $\sum a_n(x-a)^n$ can be found by letting $R = \lim |a_n/a_{n+1}|$ if this limit exists. (While this may seem to be easier to compute than the expression in the chapter, the lim sup is more likely to exist than the limit.)

3. (a) Suppose $\sum a_n(x-a)^n$ has radius of convergence R. If S is a compact subset of $(a-R, a+R)$, show that $\sum a_n(x-a)^n$ converges uniformly on S.

   (b) The requirement in (a) that S be compact is unnecessarily strong. Find a weaker condition on S that will give the same result.

4. Find the radius of convergence of the power series $\sum a_n x^n$, where the coefficients a_n are given by:

$$a_n = \begin{cases} \left(\dfrac{1}{2} - \dfrac{1}{n}\right)^n & \text{if } n = 1, 4, 7, \ldots \\ \left(\dfrac{1}{4} + \dfrac{1}{n}\right)^n & \text{if } n = 2, 5, 8, \ldots \\ \left(\dfrac{3}{4} + \dfrac{1}{n^2}\right)^n & \text{if } n = 3, 6, 9, \ldots. \end{cases}$$

5. Let (a_n) be given and let $A = \lim_{x \to 1^-} \sum_{n=0}^{\infty} a_n x^n$ if this limit exists.

   (a) Show that $A = \sum_{n=0}^{\infty} a_n$ if $\sum_{n=0}^{\infty} a_n$ converges.

   (b) Give an example to show that A might exist even if the original series $\sum a_n$ diverges. (This process is called **Abel summation**. It represents another way to interpret the sum of a series. See also Exercise 13.11.2.)

Chapter 16
Differentiation

16.1 A NEW SLANT ON DERIVATIVES

Differentiation is surely the most familiar topic from calculus. It is also the topic that is usually discussed with the most rigor. Since everyone is well acquainted with the derivative as a "limit of difference quotients," we will take the opportunity to view the subject in another way. Our approach is not at all unusual, it is just not the one typically taken in elementary textbooks. It is often said that calculus is "the study of change." A review of the subject suggests that it is equally valid to view calculus as *the study of the approximation of complicated things by simpler ones.*

When we ask whether a sequence converges, we are asking how well its limiting behavior (something we can't "touch") can be approximated by its individual terms (something we can). The convergence of a series is a question how well an infinite sum can be approximated by a finite sum. Integration seems to be a process by which complicated areas are approximated by rectangles (though this, too, is an oversimplification). The ε-δ definition of continuity can be thought of (as at the beginning of Chapter 14) as a statement about the approximation of a value of a function that we might not be able to compute by a value that we can compute.

Expanding on this view of calculus, we may say that differentiation is the study of the approximation of functions by straight lines. When we think this way, we see that the connection between the derivative and the slope of the tangent line is not just a remarkably valuable by-product of the process but the fundamental issue. The definition of the derivative is not so much about limits as it is about the relationship between two graphs.

DEFINITION 16.1: The function $f : \mathbf{R} \to \mathbf{R}$ is **differentiable** at the point a if there is a number L and a function $\eta(t)$ so that $\eta(t) \to 0$ as $t \to 0$, and $f(x) - f(a) - L(x-a) = (x-a)\eta(x-a)$. Such a number L is called the **derivative of f at a** and is denoted $f'(a)$.

It is convenient to remember this definition by noting that both sides of the equation approach 0 *even after they are divided by* $(x - a)$.

NOTE WELL: On the left side of the equation in the definition, and the first time it appears on the right, the expression $(x - a)$ is *multiplied* by the things around it. On the other hand, the expression $\eta(x - a)$ represents the *evaluation of the function* $\eta(t)$ at the number $(x - a)$. It is especially important here not to confuse multiplication with evaluation of a function.

EXAMPLES 16.1: 1. Let $f(x) = x^2$. We will check that $f'(1) = 2$. Putting the function and our guess of the derivative into the definition we have

$$\begin{aligned} & f(x) - f(a) - L(x - a) \\ = \; & x^2 - 1 - 2(x - 1) \\ = \; & x^2 - 2x + 1 \\ = \; & (x - 1)(x - 1), \end{aligned}$$

which approaches zero even after it is divided by $x - 1$. [To be precise, we can say that the definition is satisfied by letting $\eta(t) = t$.]

2. The choice of the number 2 in the previous example was a particularly good one. If, for instance, we had guessed that $f'(1) = 5$, we would find:

$$\begin{aligned} & x^2 - 1 - 5(x - 1) \\ = \; & x^2 - 5x + 4 \\ = \; & (x - 1)(x - 4), \end{aligned}$$

which does approach 0 as $x \to 1$ but does *not* approach 0 after it is divided by $x - 1$ [to make this resemble the definition, we would have to let $\eta(t) = x - 3$, but this does not go to 0 as $x \to 1$].

3. Consider $f(x) = |x|$ at the point $a = 0$. Based on the appearance of its graph, we might guess that $f'(0) = 0$. But then

$$\begin{aligned} & f(x) - f(a) - L(x - a) \\ = \; & |x| - 0 - 0(x - 0) \\ = \; & |x|. \end{aligned}$$

The definition of the derivative would require that this expression approach 0 after it is divided by x, but this is not the case since $\lim_{x\to 0} |x|/x$ does not exist (even though $\lim_{x\to 0} |x| = 0$). Consequently, the derivative of f at $a = 0$ is not 0. A similar argument can be used to show that the

derivative of f doesn't exist at all for $a = 0$.

We might feel that, in order to find a derivative, we had to know the answer beforehand, while the usual process of finding limits seems to give us the answer. In a way this is true, but remember that to find a limit by the definition we also must know the answer beforehand.

EXERCISES 16.1

1. Show that the derivative of a straight line is its slope. (Think of a short, simple answer!)

2. (a) If $f(x) = x^3$, use the definition to show that $f'(1) = 3$.

   (b) If $f(x) = x^2$, use the definition to show that $f'(a) = 2a$ for all a.

3. (a) A function f is said to have a "proper local maximum" at the point a if there is a neighborhood U of a so that $f(x) < f(a)$ for $x \in U \setminus \{a\}$. If f is differentiable at a and has a proper local maximum there, show that $f'(a) = 0$.

   (b) Prove the First Derivative test from calculus: If the continuous function f has a critical point at a [that is, either $f'(a) = 0$ or f is not differentiable at a] and there is an $\varepsilon > 0$ so that $f'(x) > 0$ for $x \in (a - \varepsilon, a)$ and $f'(x) < 0$ for $(a, a + \varepsilon)$, then f has a proper local maximum at a.

   (c) Show that a function can have only countably many proper local maxima.

4. (a) Suppose $f(x) \leq g(x) \leq h(x)$ for x in some neighborhood of a, $f(a) = h(a)$, f and h are differentiable at a, and $f'(a) = h'(a)$. Show that g is differentiable at a and that $g'(a) = f'(a)$.

   (b) Let $D(x)$ be the Dirichlet function [$D(x) = 1$ if $x \in \mathbf{Q}$ and $D(x) = 0$ if $x \notin \mathbf{Q}$]. Show that $x^2 D(x)$ is differentiable at $x = 0$.

   (c) Show that $f'(a) > 0$ does *not* imply that f is increasing in any neighborhood of a.

   (d) Doesn't (c) contradict a well-known fact from calculus?

   (e) Show that the other conditions on f and h in (a) imply that $f'(a) = h'(a)$.

5. Suppose f is differentiable at a and let $L(x) = f(a) + A(x - a)$.

   (a) If $A > f'(a)$, show that there is a $\delta > 0$ so that $L(x) < f(x)$ for $x \in (a - \delta, a)$ and $L(x) > f(x)$ for $x \in (a, a + \delta)$ and a similar result if $A < f'(a)$.

(b) Use this result to explain (yet again) why $f(x) = |x|$ is not differentiable at $a = 0$.

(c) Use this result to explain how it can be that the tangent line of $f(x) = x^3$ can cross the graph.

(d) Discuss how this property might be used to define the derivative.

(e) Is there a similar result concerning second derivatives? Consider the parabola $P(x) = f(a) + f'(a)(x - a) + (B/2)(x - a)^2$. What can be said if $B > f''(a)$?

6. One of the great advantages in looking at derivatives as we have done in this section (rather than as limits of difference quotients) is that our definition can be used, pretty much as is, for vector functions. If the inputs of our function are vectors, we can't divide by the expression $x - a$, but we can make sense of the construction we have used here. First notice that multiplying by L is a linear function from $\mathbf{R}$ to $\mathbf{R}$, then ...

DEFINITION: Let $f : \mathbf{R}^2 \to \mathbf{R}$. Then f is **differentiable** at the point (a, b) if there is a linear function $L : \mathbf{R}^2 \to \mathbf{R}$ and a function $\eta(t)$ so that $\eta(t) \to 0$ as $t \to 0$, and

$$f(x, y) - f(a, b) - L(x - a, y - b) = (x - a, y - b)\eta(\|(x - a, y - b)\|).$$

(Notice that the occurrence of L in this definition is a function evaluation rather than a multiplication).

(a) Which aspects of the vector space structure of $\mathbf{R}^2$ are needed to make sense of this definition?

(b) Considering (a), restate the definition even more generally.

(c) Show that every linear function $L : \mathbf{R}^2 \to \mathbf{R}$ can be written as a dot product with some vector; that is, for any such L, there is an ordered pair (A, B) so that $L(x, y) = (A, B) \bullet (x, y)$ (you may have already proved this in linear algebra).

(d) Suppose that $f : \mathbf{R}^2 \to \mathbf{R}$ has a derivative at (a, b). Show that the partial derivatives f_x and f_y, both exist at (a, b), and that the derivative of f at (a, b) is $(f_x(a, b), f_y(a, b))$.

(e) Suppose that both partial derivatives of a function $f : \mathbf{R}^2 \to \mathbf{R}$ exist. Is the function necessarily differentiable?

(f) Discuss the derivative of a function $f : \mathbf{R}^n \to \mathbf{R}^m$. How would it be defined? What form would the derivative take?

16.2 ORDER OF MAGNITUDE ESTIMATES

The definition of the derivative is a statement about how quickly something gets small (the expressions in the definition "get small faster than $x - a$"). It will be useful to have a way of measuring such phenomena. These notational conventions are called "Landau symbols." Their use will simplify many calculations. The relationships they symbolize are sometimes called "order of magnitude estimates."

DEFINITION 16.2: (a) We write $f = \mathrm{O}(g)$ as $x \to a$ if there are positive numbers δ and M so that $|f(x)| < M|g(x)|$ whenever $|x-a| < \delta$. This is pronounced "***f* is big oh of *g*.**"

(b) We write $f = \mathrm{o}(g)$ as $x \to a$ if for every $\varepsilon > 0$ there is a $\delta > 0$ so that $|f(x)| < \varepsilon|g(x)|$ when $|x - a| < \delta$. We say "***f* is little oh of *g*.**"

(c) If $\lim_{x\to a} f/g = 1$, we say f and g are **asymptotic** and write $f \sim g$. (We will have little occasion to use this symbol.)

The notation "as $x \to a$" is usually omitted if it is clear from the context. Note that $f = \mathrm{O}(g)$ if and only if $f(x)/g(x)$ is bounded on some interval $(a - \delta, a + \delta)$, and $f = \mathrm{o}(g)$ if and only if $\lim_{x\to a} f(x)/g(x) = 0$. One also can make order of magnitude estimates as the variable approaches infinity by replacing the phrase "$\exists\delta \ldots |x - a| < \delta$" in the definition with "$\exists B \ldots x > B$." You will check the following statements in Exercise 16.2.1.

$$x^2 = \mathrm{o}(x) \text{ as } x \to 0$$
$$\sin(x) \sim x \text{ as } x \to 0$$
$$x^3 = \mathrm{O}((7x^5 - 172x^3)/(x^2 + 14x)) \text{ as } x \to \infty$$
$$e^x \neq \mathrm{O}(x^{13}) \text{ as } x \to \infty.$$

Here is one result describing functions related by order of magnitude estimates. Its simple proof, and others like it, are left as Exercise 16.2.6.

THEOREM 16.3: *If $f = \mathrm{o}(g)$ as $x \to a$ and $h = \mathrm{o}(g)$ as $x \to a$, then $f + h = \mathrm{o}(g)$ and $f - h = \mathrm{o}(g)$ as $x \to a$.* ∎

This result can be abbreviated: "$\mathrm{o} \pm \mathrm{o} = \mathrm{o}$." Let's get back to derivatives. Using the Landau symbols, the definition can be restated:

f is differentiable at a and $f'(a) = L$ if and only if
$$f(x) - f(a) - L(x - a) = \mathrm{o}(x - a).$$

The Landau symbols give a neat appearance to our work. We will soon see that their use also has real benefit. This is a wonderful example of

a situation where nothing deeper than a choice of notation can greatly clarify a subject. Perhaps the choice of notation is "deep" after all!

EXERCISES 16.2

1. (a) Show $f = \mathrm{O}(g)$ as $x \to a$ if and only if $f(x)/g(x)$ is bounded on some interval $(a - \delta, a + \delta)$, and $f = \mathrm{o}(g)$ as $x \to a$ if and only if $\lim_{x \to a} f(x)/g(x) = 0$.

   (b) Check the statements following Definition 17.2.

2. (a) If K is a constant, show that $K(x-a) = \mathrm{o}(x-a)$ if and only if $K = 0$.

   (b) Show that $K(x-a) = \mathrm{O}(x-a)$ for any K.

3. The statement "If $f = \mathrm{o}(g)$ then $f = \mathrm{O}(g)$" can be abbreviated: $\mathrm{o} \Rightarrow \mathrm{O}$. Prove this.

4. (a) Show that $f = \mathrm{o}(1)$ as $x \to a$ if and only if $f(x) \to 0$ as $x \to a$.

   (b) Show that $f = \mathrm{O}(1)$ as $x \to a$ if and only if $f(x)$ is bounded on some neighborhood of a.

   (c) Show that if $f \sim g$, then $f = \mathrm{O}(g)$ and $g = \mathrm{O}(f)$.

   (d) Show that the converse of (c) is not true.

5. Show that if $f(t) = \mathrm{o}(g(t))$ as $t \to a$, and $h(t) \to a$ as $t \to b$, then $f \circ h(t) = \mathrm{o}(g \circ h(t))$ as $t \to b$.

6. (a) Interpret and prove:
   (i) $\mathrm{o} + \mathrm{o} = \mathrm{o}$
   (ii) $\mathrm{o} + \mathrm{O} = \mathrm{O}$
   (iii) $\mathrm{O} + \mathrm{O} = \mathrm{O}$
   (iv) $\mathrm{o}(\mathrm{O}) = \mathrm{o}$ [(iv) and (v) represent compositions of functions]
   (v) $\mathrm{O}(\mathrm{o}) = \mathrm{o}$
   (vi) $\mathrm{o} \times \mathrm{O} = \mathrm{o}$
   (vii) $\mathrm{o} \times \mathrm{o} = \mathrm{o}$

   (b) What can be said about $\mathrm{O} \times \mathrm{O}$?

16.3 BASIC DIFFERENTIATION THEOREMS

We have shown that the derivative of x^2 at $x = 1$ is 2 and not 5. Does the definition *exclude* the possibility that some number other than 2 could work? Yes:

THEOREM 16.4: *A function can have only one derivative at a point.*

PROOF: Suppose it happens both that $f(x)-f(a)-L_1(x-a) = o(x-a)$ and that $f(x) - f(a) - L_2(x - a) = o(x - a)$. Then, by Theorem 16.3,

$$[f(x) - f(a) - L_1(x - a)] - [f(x) - f(a) - L_2(x - a)] = o(x - a).$$

Now the left side of this equality is $(L_2 - L_1)(x - a)$, and this can be $o(x - a)$ only if $L_2 = L_1$, by Exercise 16.2.2.a. ∎

With Theorem 16.4 in hand, we can prove the following reassuring result.

THEOREM 16.5: *The function f is differentiable at a if and only if*

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. If so, this limit is $f'(a)$.

PROOF: This proof is made easier because we know what the derivative should be. First, suppose f is differentiable at a and that $f'(a) = L$. We show that the limit exists by showing that it is equal to L:

$$\begin{aligned} & \lim_{x \to a} \left| \frac{f(x) - f(a)}{x - a} - L \right| \\ = & \lim_{x \to a} \frac{|f(x) - f(a) - L(x - a)|}{|x - a|} \\ = & \ 0 \end{aligned}$$

since the numerator of the fraction in the second line is $o(x - a)$.

Now suppose $\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = L$. Here we need to show that $f(x) - f(a) - L(x - a) = o(x - a)$. By Theorem 16.4, this will tell us that $L = f'(a)$. Let $\eta(t) = \frac{f(a+t)-f(a)}{t} - L$ (note that $\eta(t) \to 0$ as $t \to 0$). Then

$$\begin{aligned} & f(x) - f(a) - L(x - a) \\ = & \ f(x) - f(a) - \left[\frac{f(x) - f(a)}{x - a} - \eta(x - a) \right] (x - a) \\ = & \ \eta(x - a)(x - a) \\ = & \ o(x - a). \ ∎ \end{aligned}$$

With Theorem 16.5, we may prove the following result in the usual way.

THEOREM 16.6: *If f is differentiable at a, it is continuous at a.* ∎

Theorem 16.5 returns differentiation to a familiar setting, and the proofs of the basic theorems of differentiation can be referred back to a calculus course. Let us investigate some of those results in this new setting, though.

THEOREM 16.7: *Suppose f and g are differentiable at a and $c \in \mathbf{R}$, then*

(a) *$f + g$ is differentiable at a, and $(f+g)'(a) = f'(a) + g'(a)$.*

(b) *cf is differentiable at a, and $(cf)'(a) = cf'(a)$.*

(c) *fg is differentiable at a, and $(fg)'(a) = f(a)g'(a) + f'(a)g(a)$.*

(d) *f/g is differentiable at a, and*

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}$$

[*as long as there is a neighborhood of a in which $g(x) \neq 0$*].

PROOF: We will prove only part (c), leaving the rest as Exercise 16.3.2. In view of Theorem 16.4, we need only show that

$$f(x)g(x) - f(a)g(a) - [f(a)g'(a) + f'(a)g(a)](x-a) = o(x-a).$$

We use the trick of adding and subtracting something on the left side of the equation [here it will be $f(a)g(x)$] and rearranging terms, we find:

$$\begin{aligned} & f(x)g(x) - f(a)g(a) - [f(a)g'(a) + f'(a)g(a)](x-a) \\ = \;& [f(x) - f(a)]g(x) - f'(a)g(a)(x-a) + \\ & \qquad\qquad [g(x) - g(a)]f(a) - g'(a)f(a)(x-a). \end{aligned}$$

The last two terms combine to become $f(a)[g(x) - g(a) - g'(a)(x-a)]$, which is $o(x-a)$ regardless of the value of $f(a)$ since g is differentiable at a. We can't dispose of the first two terms so easily. How is $g(x)$ related to $g(a)$? Since g is differentiable at a, $g(x) = g(a) + g'(a)(x-a) + o(x-a)$. Inserting this observation into the first term, we have

$$\begin{aligned} & [f(x) - f(a)]g(x) - f'(a)g(a)(x-a) \\ = \;& [f(x) - f(a)][g(a) + g'(a)(x-a) + o(x-a)] - f'(a)g(a)(x-a) \\ = \;& [f(x) - f(a) - f'(a)(x-a)]g(a) + \\ & \qquad\qquad [f(x) - f(a)]g'(a)(x-a) + [f(x) - f(a)]o(x-a). \end{aligned}$$

The first of these terms is $o(x-a)$ since f is differentiable at a. Since f

is continuous at a, $f(x) - f(a) \to 0$ as $x \to a$. Thus the second and third terms are also $o(x - a)$. We have shown that

$$\begin{aligned} & f(x)g(x) - f(a)g(a) - [f(a)g'(a) + f'(a)g(a)](x-a) \\ = \; & o(x-a) + o(x-a) + o(x-a) + o(x-a) \\ = \; & o(x-a), \end{aligned}$$

and we are done. ∎

This proof shows the usefulness of the Landau notation. As each part of the sum is shown to be the "right size"—here this means $o(x - a)$—we can pretty much set it aside. The specific content of these expressions is no longer important. If we were to do this proof using the standard definition, we would have to keep track of these things more carefully. More importantly, the notation allows us to deal with *equalities* rather than estimates and inequalities. In this proof we have used statements like "(something that goes to 0)$\times(x - a) = o(x - a)$," which you verified in Exercise 16.2.6.

We will prove one more standard theorem from calculus. The Chain rule, certainly the most important differentiation formula, is treated badly in some calculus texts. The "obvious" proof, actually presented in some texts, is simply incorrect (though it can be repaired). You will examine and fix it in Exercise 16.3.3.

THEOREM 16.8: *If g is differentiable at a and f is differentiable at $g(a)$, then the composition $f \circ g$ is differentiable at a and*

$$(f \circ g)'(a) = f'(g(a))g'(a).$$

PROOF: We need to show that

$$(f \circ g)(x) - (f \circ g)(a) - f'(g(a))g'(a)(x - a) = o(x - a).$$

Here we must be a bit clever. In the expression $f(g(x)) - f(g(a))$, we have $g(x)$ and $g(a)$ inserted into f where we would like to see x and a. But we can get around this problem. Since g is continuous at a, we know that $g(x) - g(a) \to 0$ as $x \to a$. We may replace x and a in the definition with $g(x)$ and $g(a)$ (see Exercise 16.2.6), to obtain

$$f(g(x)) - f(g(a)) - f'(g(a))[g(x) - g(a)] = o(g(x) - g(a)).$$

Since g is differentiable at a, we have $g(x) - g(a) = g'(a)(x - a) + o(x - a)$. Plugging this in to the last equality gives us

$$\begin{aligned} & f(g(x)) - f(g(a)) - f'(g(a))[g'(a)(x-a) + o(x-a)] \\ = \; & o(g'(a)(x-a) + o(x-a)) \end{aligned}$$

or

$$\begin{aligned}
& f(g(x)) - f(g(a)) - f'(g(a))g'(a)(x-a) \\
= \; & o(g'(a)(x-a) + o(x-a)) + f'(g(a))o(x-a).
\end{aligned}$$

The last term is $o(x-a)$. We can simplify the first term by observing that $g'(a)(x-a) = O(x-a)$. You proved in Exercise 16.2.6 statements like $O + o = O$ and $o(O) = o$, which together give us the result. ∎

In this proof, the specific content of the expression $g'(a)(x-a)$ was needed at one stage (on the left side of the last equation), while at another we only needed to know that $g'(a)(x-a) = O(x-a)$.

EXERCISES 16.3

1. (a) Prove Theorem 16.6 in the usual way.

   (b) If $f(x) - f(a) = O(x-a)$, show that f is continuous at a.

   (c) Prove Theorem 16.6 using the definition of differentiability given in the chapter.

   (d) Give an example of a function f and a point a such that f is continuous at a but it is *not* the case that $f(x) - f(a) = O(x-a)$.

2. Complete the proof of Theorem 16.7.

3. (a) What is wrong with the following "proof" of the chain rule?

   (I) Note that $\dfrac{f(g(x)) - f(g(a))}{x-a} = \dfrac{f(g(x)) - f(g(a))}{g(x) - g(a)} \left(\dfrac{g(x) - g(a)}{x-a} \right)$.

   (II) The middle fraction approaches $f'(g(a))$, while the one on the right approaches $g'(a)$, so the whole thing approaches $f'(g(a))g'(a)$.

   (b) Find a way to use this approach in a valid proof.

4. (a) If $f(x) = x$, show that $f'(a) = 1$ for all a.

   (b) If $f(x) = x^n$, $n \in \mathbf{N}$, show that $f'(a) = na^{n-1}$ for any a.

   (c) Let $f(x) = 1/x$ and $a > 0$. Show that $f'(a) = -1/a^2$.

5. Suppose f and g each have n derivatives at a point a. Find a formula for the nth derivative of the product fg at a (this is called **Leibniz' Rule**).

6. (a) Suppose that f is differentiable at a point a. Show that

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a-h)}{2h}.$$

(b) Draw a picture that describes the fraction on the right.

(c) The expression on the right in (a) is called the **symmetric derivative** of f at a. Give an example of a function that has a symmetric derivative at a point but is not differentiable there.

(d) (Difficult!) At how many points can a function have a symmetric derivative but not a derivative?

16.4 THE MEAN VALUE THEOREM

The Mean Value theorem gives theoretical muscle to virtually all the important results of calculus. In this section we examine the collection of results leading up to the Mean Value theorem. The road to the Mean Value theorem is not really very long, and the road itself was pretty well constructed in calculus. All we need to do now is to find the entrance ramp. We will go a bit out of our way to use our definition of the derivative instead of limit quotients, but we will arrive in familiar territory.

We highlight the first lemma in part because it is the main step in the one that follows it, and because it brings us perilously close to jumping to a bad conclusion. We might try to say something like "if $f'(a) > 0$, then f is increasing at a." But the phrase "increasing at a" is not precisely defined, and the local behavior of a function whose derivative is positive at a single point may not be what we expect (see Exercise 16.1.4).

LEMMA 16.9: *Suppose f is differentiable at a and that $f'(a) \neq 0$. There is an $\varepsilon > 0$ so that if $x_1 \in (a - \varepsilon, a)$ and $x_2 \in (a, a + \varepsilon)$, then one of $f(x_1)$ and $f(x_2)$ is greater than $f(a)$ and the other is less than $f(a)$.*

PROOF: We must take advantage of the approximation of a function by its tangent line. Let us assume $f'(a) > 0$. We will show that there is an $\varepsilon > 0$ so that $f(b) > f(a)$ for all $b \in (a, a + \varepsilon)$ (the other part of the proof is similar). By the definition of the derivative, we know that $f(x) = f(a) + f'(a)(x - a) + \eta(x - a)(x - a)$ and so, for *any* $b > a$,

$$\begin{aligned} & f(b) - f(a) \\ = \quad & f'(a)(b-a) + \eta(b-a)(b-a) \\ = \quad & \left[f'(a) + \eta(b-a)\right](b-a) \end{aligned}$$

We want $f(b) - f(a) > 0$. Since $f'(a) > 0$ and $b - a > 0$, we need only be sure that $|\eta(b-a)| < f'(a)$ for this to be true. But $\eta(b-a) \to 0$ as $b \to a$, and so we may pick ε so that $|\eta(b-a)| < f'(a)$ whenever $b \in (a, a+\varepsilon)$. ∎

Recall that a function f has a **local maximum** at a [and we say $f(a)$

is a local maximum] if there is an $\varepsilon > 0$ such that $f(x) \leq f(a)$ for any $x \in (a-\varepsilon, a+\varepsilon)$. **Local minimum** is defined similarly, and a **local extreme** is a number that is either a local maximum or minimum. Lemma 16.10 is a corollary to Lemma 16.9, but we call it a lemma because of the role it plays in what follows. The proof is left as Exercise 16.4.2.

LEMMA 16.10: *If f is differentiable at a and f has a local extreme at a, then $f'(a) = 0$.* ∎

THEOREM 16.11: (Rolle's Theorem) *Suppose $f : [a, b] \to \mathbf{R}$ is continuous, that f is differentiable at each point of (a, b), and that $f(a) = f(b) = 0$. Then there is a point $c \in (a, b)$ with $f'(c) = 0$.*

PROOF: By the Extreme Value theorem, f has a maximum and a minimum on $[a, b]$. If both these values are 0, then $f(x) = 0$ for all x, and so $f'(x) = 0$ for all x. Any element of (a, b) will serve as the point c. Now suppose that the maximum value of f is $f(c)$ and $f(c) > 0$ (a similar argument applies we know only that the minimum is negative). Since $f(c) > 0$, c is neither a nor b, and by hypothesis f is differentiable at c. Now $f(c)$ is also a local maximum, and so by Lemma 16.10, $f'(c) = 0$. ∎

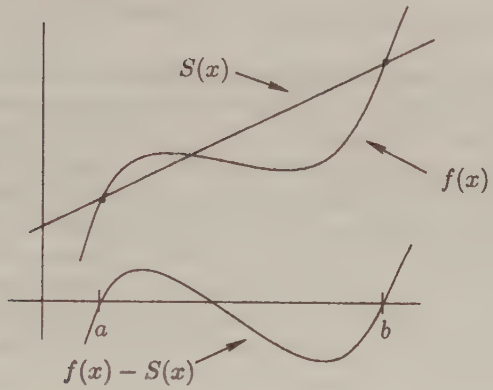
Notice that the condition $f(a) = f(b) = 0$ can easily be replaced by the condition $f(a) = f(b)$. The completeness of the real numbers enters this proof in a small but crucial way. Extreme values of a function are the only ones for which we can guarantee that $f' = 0$ (though it might happen that $f' = 0$ at other points), and the completeness of the real numbers guarantees the existence of extreme values. Rolle's theorem is not true for functions whose domains are intervals of rational numbers.

THEOREM 16.12: (The Mean Value Theorem) *Suppose $f : [a, b] \to \mathbf{R}$ is continuous and that f is differentiable at each point of (a, b). There is a point $c \in (a, b)$ with $f'(c) = (f(b) - f(a))/(b - a)$.*

PROOF: This is one of those rare proofs that can be seen almost entirely with a picture (below). We subtract from f the secant line through $(a, f(a))$ and $(b, f(b))$, which we call $S(x)$. Note that the slope of S is $[f(b) - f(a)]/(b - a)$. Since $F(x) = f(x) - S(x)$ satisfies the hypotheses of Rolle's theorem (be sure to check this), there is a real number $c \in (a, b)$ with $F'(c) = 0$ (in this picture there are two such points). But

$$F'(c) = f'(c) - S'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0,$$

which gives us what we want. ∎



EXERCISES 16.4

1. Complete the proof of Lemma 16.9.

2. Prove Lemma 16.10.

3. Complete the proof of Theorem 16.11.

4. Give an example to show that Rolle's theorem fails for functions defined on the rational numbers.

5. (a) Prove the basic theorem of elementary calculus: *If* $f : [a, b] \to \mathbf{R}$ *is continuous, then* (1) f *attains a maximum on* $[a, b]$ *and* (2) *that maximum occurs at* (i) *a point where* $f'(x) = 0$, (ii) *a point where* $f'(x)$ *does not exists, or* (iii) a *or* b.

   (b) Explain why it is not really necessary (though every calculus text does it) to single out a and b as possible solutions in part (a).

6. (a) Let $f : [a, b] \to \mathbf{R}$ and $g : [a, b] \to \mathbf{R}$ both satisfy the hypotheses of the Mean Value theorem. By considering the function

   $F(x) = (f(x) - f(a))(g(b) - g(a)) - (g(x) - g(a))(f(b) - f(a)),$

   show that there is a number $c \in (a, b)$ such that

   $$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

   (This is called the **Cauchy Mean Value theorem**.)

   (b) By considering f and g to be the coordinate functions of a parametric curve, interpret the Cauchy Mean Value theorem geometrically.

   (c) Use the Cauchy Mean Value theorem to prove l'Hôpital's rule.

16.5 THE MEANING OF THE MEAN VALUE THEOREM

The Mean Value theorem is one of the most important in calculus. This stems both from the nature of the theorem itself and the things that can be done with it. The Mean Value theorem is the first one we encounter in calculus that gives us something specific (the point c). Earlier theorems are more negative in tone ("There are no differentiable functions that are not continuous.") But the real power of the Mean Value theorem lies in what can be done with it. The diagram in Chapter 10 reminds us that the Mean Value theorem is the bridge to both Taylor's theorem (which we will prove in the next section) and the Fundamental theorem (which we will prove in in the next chapter). The first, and most transparent, uses of the Mean Value theorem in calculus are in the following theorem.

THEOREM 16.13: (a) *If $f : (a, b) \to \mathbf{R}$ is differentiable and $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b).*

(b) *If $f : (a, b) \to \mathbf{R}$ is differentiable and $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on (a, b)*

(c) *If $f : (a, b) \to \mathbf{R}$ is differentiable and $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on (a, b).*

PROOF: We will prove (a); the rest is left as Exercise 16.5.1. Suppose f is not constant. Then there are points $x, y \in (a, b)$, with $f(x) \neq f(y)$. Then $[f(y) - f(x)]/(y - x) \neq 0$. By the Mean Value theorem, there is a point c between x and y with $[f(y) - f(x)]/(y - x) = f'(c) \neq 0$, a contradiction. ∎

EXERCISES 16.5

1. Complete the proof of Theorem 16.13.

2. Suppose that f and g are differentiable on an open interval (a, b) and that $f'(x) = g'(x)$ for all $x \in (a, b)$. Show that $f(x) - g(x)$ is constant on (a, b). Where does this come up in elementary calculus?

3. (a) Suppose f has the property that, for any x and y in the domain of f, $|f(x) - f(y)| \leq |x - y|$. Show that f is uniformly continuous.

   (b) Suppose f has the property that there is a number $\alpha > 1$ so that, for any x and y in the domain of f, $|f(x) - f(y)| \leq |x - y|^\alpha$. Show that f is constant. (Hint: Show that the derivative of f is always 0.)

4. By considering the derivative of the quotient $f(x)/e^x$, show that if $f'(x) = f(x)$ for all x and $f(0) = 1$, then $f(x) = e^x$.

16.6 TAYLOR POLYNOMIALS

When we approximate a function by its tangent line, we suffer an error that is $o(x - a)$. Perhaps we can get an error that is, say, $o((x - a)^2)$ by approximating the function by a (not very much) more complicated object—a polynomial of degree higher than 1. Though an error of $o(x-a)$ is enough to give us what we needed in the previous sections, it is not very delicate as a measuring device. There is much territory covered by "$o(x - a)$," and we might well desire a more precise statement of the error in some process. By a happy coincidence, we can obtain better approximations and better error estimates at the same time.

DEFINITION 16.14: If $f(a), f'(a), \ldots, f^{(n)}(a)$ all exist, the ***n*th Taylor polynomial**[1] of the function f at the point a is given by

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k$$

Here $f^{(0)}(a) = f(a)$. Notice that a Taylor polynomial represents the approximation of a complicated object (f) by a simpler one (T_n). As we get better at such processes, we can expand our idea of what is "simple." The tangent line to a function is special because it matches the values of both the function and its derivative at the point where it is computed. If we form an approximation with a polynomial, we can expect higher-order derivatives to match as well. You will show in Exercise 16.6.3 that the values of the first n derivatives of T_n and f match at a. The question remains how T_n is related to f at points other than a.

THEOREM 16.15: (Taylor's Theorem) *If $x > a$, $f^{(n)}$ is continuous on $[a, x]$, and $f^{(n+1)}$ exists on (a, x), there is a point $c \in (a, x)$ with*

$$f(x) = T_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1}.$$

(A similar statement holds if $x < a$.)

PROOF: We will view as a problem of solving an equation. If x and a are given, there is *some* number $A_n(x)$ with $f(x) = T_n(x) + A_n(x)(x-a)^{n+1}$. We want to show that $A_n(x) = f^{(n+1)}(c)/(n+1)!$ for some $c \in (a, x)$.

[1] If $a = 0$, this is called a **Maclaurin Polynomial**. This doesn't mean that Maclaurin became famous by setting $a = 0$. Among his other accomplishments, Maclaurin was the author of the first calculus text in English (originals of which can still be found in rare-book stores).

Here we resort to some trickery. Let

$$F(t) = f(t) + f'(t)(x-t) + \cdots + f^{(n)}(t)(x-t)^n/n! + A_n(x)(x-t)^{n+1}.$$

By hypothesis, $f^{(n)}$ is continuous on $[a,x]$ and $f, f', \ldots,$ and $f^{(n)}$ are all differentiable, and so F is continuous on $[a,x]$ and differentiable on (a,x). Note that $F(x) = f(x)$ [since all but the first term are 0 when we plug in $t = x$] and $F(a) = f(x)$ [by the choice of $A_n(x)$]. Thus the function of t given by $F(t) - f(x)$ is 0 at both a and x. Applying Rolle's theorem, there is a point $c \in (a,x)$ with $F'(c) = 0$. Now

$$\begin{aligned} F'(t) = f'(t) &+ f''(t)(x-t) - f'(t) + \cdots \\ &+ f^{(n+1)}(t)(x-t)^n/n! - A_n(x)(n+1)(x-t)^n. \end{aligned}$$

Most of the terms in this expression cancel, leaving

$$F'(t) = f^{(n+1)}(t)(x-t)^n/n! - A_n(x)(n+1)(x-t)^n.$$

Plugging in $t = c$, setting the left side equal to 0, and solving for $A_n(x)$ gives us the desired result. ∎

We will set $R_n(x) = f(x) - T_n(x)$. [$R_n(x)$ is the "remainder" upon approximating $f(x)$ with $T_n(x)$.] Then Taylor's theorem says there is a number c between a and x so that $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$.

EXAMPLES 16.6: 1. Let $f(x) = \sin x$ and $a = 0$. Then $T_1(x) = x$. Since $|f''(x)| \le 1$ for all x, we have $|R_1| \le |x|^2/2$. The approximation of $\sin x$ by x is thus $\mathrm{O}(x^2)$ as $x \to 0$. Since $f''(0) = 0$, we also have $T_2(x) = x$, and since it is also the case that $|f'''(x)| \le 1$, we have $|R_2| \le |x|^3/6$. The approximation by the tangent line is thus $\mathrm{O}(x^3)$, even better than expected.

2. The tangent line to a function f is T_1. For $n = 1$, Taylor's theorem says $f(x) - T_1(x) = f''(c)(x-a)^2/2$. If it happens that $f''(c)$ is bounded, then $f(x) - T_1(x) = \mathrm{O}((x-a)^2)$, and so $f(x) - T_1(x) = \mathrm{o}(x-a)$, which brings us back to the definition of the derivative. The similarity between Taylor's theorem and the definition of the derivative raises a question: Can a statement of the form of Taylor's theorem be used to *define* higher derivatives? You will explore this in Exercise 16.6.6.

EXERCISES 16.6

1. (a) Use Taylor's theorem with $n = 3$ to find an estimate for $\sqrt{65}$ [this is the function $f(x) = \sqrt{x}$ evaluated at $x = 65$] and give an estimate of the error.

(b) What degree of Maclaurin polynomial is needed to estimate $e^{0.2}$ with an error of less than 0.001?

2. (a) Use Taylor's theorem to prove the Second Derivative test.

   (b) Extend the Second Derivative test. If $f'' = 0$, are there conditions under which the conclusions can still be drawn?

3. (a) Show that T_n and f match at the point a for their first n derivatives.

   (b) Suppose $f(x)$ is a polynomial of degree n. Show *without doing any calculations* that $T_m(x) = f(x)$ for $m \geq n$.

4. Verify the statement in the proof of Taylor's theorem that "Most of the terms in this expression cancel."

5. Recall that Taylor's theorem says that, under appropriate conditions,

   $f(x) - (f(a) + f'(a)(x-a) + (f''(a)/2)(x-a)^2) = o((x-a)^2)$ as $x \to a$.

   Suppose f satisfies the hypotheses of Taylor's theorem for $n = 2$, and that $A \neq f''(a)$. Show that

   $$f(x) - (f(a) + f'(a)(x-a) + (A/2)(x-a)^2) \neq o((x-a)^2).$$

6. Discuss how the observation in Exercise 16.6.5 could be used to define the second (and higher) derivatives.

7. Describe *in words* what is going on in Example 16.6.2. Does a bound on the second derivative have implications for the shape of a graph? Does your answer describe functions such as $f(x) = x^2$, whose second derivative is constant but whose shape seems to change from place to place? Look up "curvature" in a calculus book.

16.7 TAYLOR SERIES

The form of Taylor polynomials suggests a natural question: What if we simply continue computing terms forever?

DEFINITION 16.16: If the function f has derivatives of all orders at a, the **Taylor series** for f at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(as before, this is a **Maclaurin series** if $a = 0$).

For instance, the Maclaurin series for $f(x) = e^x$ is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ since $f^{(n)}(0) = 1$ for all n. There is only one question of real interest: Is the Taylor series of a function equal to the function? The next example shows that it is possible for the answer to be no in quite dramatic fashion.

EXAMPLES 16.7: 1. Let $f(x) = e^{-1/x^2}$ if $x \neq 0$ and $f(0) = 0$. You will show in Exercise 16.7.1 that $f^{(n)}(0) = 0$ for all n. Thus the Maclaurin series for f is *identically* 0, but the function certainly is not. The series is equal to the function only at $x = 0$. This function is often referred to as "infinitely flat" at 0.

If g is any function that is equal to its Maclaurin series (such a function is called **analytic**, but we don't know yet that there are any!), and f is as above, then $(g+f)^{(n)}(0) = g^{(n)}(0)$ for all n, and so g and $g+f$ have the same Maclaurin series. But $g(x) = (g+f)(x)$ only for $x = 0$. We can't tell whether a function is analytic just by looking at its series, but the following corollary to Taylor's theorem helps us make this decision. We need only observe that the Taylor *polynomials* of a function are the partial sums of its Taylor *series* to obtain:

COROLLARY 16.17: *If $R_n(x)$ is as in Theorem 16.15 and $R_n(x) \to 0$, then*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \; . \blacksquare$$

EXAMPLES 16.7: 2. If $f(x) = e^x$, then $R_n(x) = e^c x^{n+1}/(n+1)!$. If $x > 0$, we have $e^c < e^x$, so $R_n(x) < e^x x^{n+1}/(n+1)!$. We have seen elsewhere that $\sum x^{n+1}/(n+1)!$ converges for all values of x. By the nth Term test, $\lim_{n\to\infty} x^{n+1}/(n+1)! = 0$, so $R_n(x) \to 0$ for all $x > 0$. Thus $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $x > 0$. (The conclusion is much easier to reach for $x \leq 0$.)

The hypothesis of our final corollary is quite restrictive but the result is still useful in many contexts.

COROLLARY 16.18: *If there is a number M so that $|f^{(n)}(t)| \leq M$ for all n and for all t between a and x, then f is equal to its Taylor series between a and x.* $\blacksquare$

EXERCISES 16.7

1. Let $f(x) = e^{-1/x^2}$ if $x \neq 0$ and $f(0) = 0$. Show that $f^{(n)}(0) = 0$ for

all n.

2. (a) Find the Maclaurin series for $\sin x$ and $\cos x$ and show that they converge for all x to their respective functions.

   (b) Notice that, if you were to change all of the signs in the series for $\sin x$ and $\cos x$ to $+$ and add the results together, you would get the series for e^x. Does this mean anything?

3. (a) Find the Maclaurin series for $\ln(1+x)$ and discuss its convergence.

   (b) True or False: The alternating harmonic series converges to $\ln(2)$.

4. (a) Use the result of Exercise 4.5.17 to expand $\left(1+\frac{x}{n}\right)^n$.

   (b) Recall from calculus that the limit of this sequence is e^x. Compare the result in (a) to the Maclaurin series $e^x = 1+x+x^2/2+x^3/6+\cdots$.

5. (a) If we let $f(x) = e^x$, then the "infinitely flat" function is (essentially) $f(-1/x^2)$. Notice that $f(x)$ has a horizontal asymptote as $x \to -\infty$. Show that this is not enough to make $f(-1/x^2)$ infinitely flat by considering $f(x) = \frac{1}{1+x^2}$.

   (b) Find another infinitely flat function.

   (c) What aspect of the behavior of e^x as $x \to -\infty$ *does* make e^{-1/x^2} infinitely flat?

6. (a) Construct a function that is infinitely differentiable everywhere and is infinitely flat at more than one point.

   (b) Construct a function that is infinitely differentiable everywhere and is infinitely flat at each integer.

   (c) (Research!) At how many points can an infinitely differentiable function be infinitely flat and yet not be constant?

7. On the graph of a typical parabola, a secant line connecting two points on the curve lies entirely above the curve, like this:

A graph with this property is said to be **convex**.

(a) Show that if f is convex and differentiable, then any *tangent* line to f lies entirely *below* the graph.

(b) Show that if f is convex and differentiable, then f' is increasing.

(c) Show that if f is convex and twice differentiable, then f'' is non-negative.

(d) If f is convex (differentiable or not), and $a < b < c < d$ show that

$$\frac{f(b)-f(a)}{b-a} \leq \frac{f(d)-f(c)}{d-c}$$

Interpret this in words.

8. (a) Give an example of a discontinuous convex function $f : [0,1] \to \mathbf{R}$.

   (b) Show that a convex function whose domain is an open interval must be continuous.

   (c) Show that if f is known to be continuous, it is sufficient to require that the *midpoint* of any segment connecting points on the curve is above the curve for f to be convex.

9. A region in the plane is called **convex** if the line segment connecting any two points in the region lies entirely within the region.

   (a) Draw a picture to illustrate this.

   (b) Show that a *function* is convex if and only if the *region* above its graph is convex (note that "convex" has two different meanings in this sentence).

   (c) Is the union (or intersection) of two convex regions necessarily convex?

   (d) Explain why this definition of convex makes sense for regions in any vector space.

   (e) Does the definition of a convex function make sense for functions whose domains or ranges are vector spaces?

   (f) Which subsets of the real line are convex?

10. A region in the plane is called **star convex** if it contains a point p so that the line segment connecting any point of the region to p lies entirely within the region.

    (a) Draw a picture to illustrate why this is an appropriate name for such a region.

    (b) Show that any convex region is also star convex.

11. (a) If V is a vector space and $\|\bullet\|$ is a norm on V (see Exercise 15.2.9), show that the set $B = \{v \in V : \|v\| \leq 1\}$ is convex (B is called the **unit ball in** V).

(b) Show that B has the property: $(v \in B) \Leftrightarrow (-v \in B)$.

(c) A set with the property in (b) is called **balanced**. Say why this is an appropriate term.

(d) Show that B has the property: $\forall v \in V \exists k \in \mathbf{R} \ni (kv \in B)$.

(e) A set with the property in (b) is called **absorbing**. Say why this is an appropriate term.

(f) Draw pictures to show that the properties "convex," "balanced," and "absorbing" are independent. (You will need to draw three pictures, each of which is a set having two of the properties but not the third.)

(g) Suppose C is a subset of a vector space V that is closed, convex, balanced, and absorbing. For $v \in V$, let $\mu_C(v) = \inf\{k > 0 : \frac{1}{k}v \in C\}$. This is called the **Minkowski functional** of C. Show that μ_C is a norm on V and that C is its unit ball.

12. (a) If $\varphi : [c, d] \to \mathbf{R}$ is a convex function and $x_1, x_2, \ldots, x_n \in [c, d]$, show that

$$\varphi\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \leq \frac{\varphi(x_1) + \varphi(x_2) + \cdots + \varphi(x_n)}{n}.$$

(b) If φ is as in (a) and $f : [0, 1] \to [c, d]$, show **Jensen's Inequality**:

$$\varphi\left(\int_0^1 f(x)dx\right) \leq \int_0^1 \varphi(f(x))dx.$$

(c) What is the significance of the fact that the domain of f in part (b) is $[0, 1]$? Modify the result so that this is not necessary.

(d) Show that $\left(\int_0^1 \sin(x)\, dx\right)^2 \leq \int_0^1 \sin^2 x\, dx$.

Chapter 17

Integration

17.1 UPPER AND LOWER RIEMANN INTEGRALS

In calculus we learned to compute Riemann integrals and examined some of their uses. While techniques and applications of integration are a major part of a calculus course, these are not our concern here. Our goal now is to describe carefully the limit process involved in the definition of the integral and establish some of the major theorems of the subject. As we did in Chapter 16, we will adopt a slightly different approach from that usually taken in calculus texts. This method, which highlights the role of the completeness of the real numbers and smoothes out many of the proofs, is accessible to us now thanks to our knowledge of infima and suprema and our familiarity with the Cauchy criterion. The idea of examining the integral through upper and lower estimates was developed by Darboux about 20 years after Riemann's original work. In the setting of the real numbers, however, the results are the same (Theorem 17.18), and the process is still usually called "Riemann integration," although "Riemann-Darboux integration" would be more appropriate (and some people do use this name). Since we are familiar with the major characters in this story, we can jump right in.

DEFINITION 17.1: (a) A **partition** of the closed interval $[a, b]$ is a set $P = \{x_0, x_1, \ldots, x_n\}$ with $a = x_0 < x_1 < \ldots < x_n = b$. The intervals $[x_{k-1}, x_k]$ are called the **subintervals** of $[a, b]$ given by P.

(b) A partition P' is a **refinement** of P if $P \subseteq P'$.

DEFINITION 17.2: If f is bounded and $P = \{x_0, x_1, \ldots, x_n\}$, let $M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$, and $m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$, $k = 1, \ldots, n$. Then the **upper and lower Riemann sums** for f over $[a, b]$ with partition P are given by $U(f, P) = \sum_{k=1}^{n} M_k \Delta x_k$ and $L(f, P) = \sum_{k=1}^{n} m_k \Delta x_k$, respectively (where $\Delta x_k = x_k - x_{k-1}$).

Since $m_k \leq M_k$ for all k (Exercise 5.1.4), it is clear that $L(f, P) \leq U(f, P)$

for any partition P. But this inequality between upper and lower sums holds in an even stronger sense.

LEMMA 17.3: (a) *If P' is a refinement of P, then $L(f,P) \le L(f,P')$ and $U(f,P') \le U(f,P)$.*

(b) *If P_1 and P_2 are any two partitions, $L(f,P_1) \le U(f,P_2)$.*

PROOF: (a) Partitions are, by definition, finite. We will prove this is true when P' is obtained by adding *one* point to P. The result will follow by induction. Suppose $P' = P \cup \{x^*\}$ and $x_{k-1} < x^* < x_k$. The contributions to the lower sums constructed with P and P' are the same from all intervals except $[x_{k-1}, x_k]$, and

$$\begin{aligned}
& [\inf_{x \in [x_{k-1},x_k]} f(x)](x_k - x_{k-1}) \\
= \; & [\inf_{x \in [x_{k-1},x_k]} f(x)](x_k - x^*) + [\inf_{x \in [x_{k-1},x_k]} f(x)](x^* - x_{k-1}) \\
\le \; & [\inf_{x \in [x_{k-1},x^*]} f(x)](x_k - x^*) + [\inf_{x \in [x^*,x_k]} f(x)](x^* - x_{k-1}).
\end{aligned}$$

The last inequality holds by Exercise 5.1.6. It follows that $L(f,P) \le L(f, P \cup \{x^*\})$. Upper sums are handled similarly.

(b) Let $P = P_1 \cup P_2$. This is called the **common refinement** of P_1 and P_2. Then by part (a) and the observation before the lemma,

$$L(f,P_1) \le L(f,P) \le U(f,P) \le U(f,P_2). \blacksquare$$

Lemma 17.3 confirms the impression given by the familiar pictures used to describe the integral in calculus, that making a refinement of a partition makes any upper sum smaller and any lower sum larger. More importantly, Lemma 17.3 tells us that, as long as the function f is bounded, the collection of lower sums is bounded above (by any of the upper sums) and the collection of upper sums is bounded below (by any of the lower sums). The Least Upper Bound property then allows us to make the following definition.

DEFINITION 17.4: If $f : [a,b] \to \mathbf{R}$ is bounded, then the **upper and lower Riemann integrals** for f over $[a,b]$ are $U(f) = \inf U(f,P)$ and $L(f) = \sup L(f,P)$, respectively, where the supremum and infimum are taken over all partitions P of $[a,b]$.

The next theorem follows from Lemma 17.3 and Exercise 5.1.11.

THEOREM 17.5: $L(f) \le U(f)$. $\blacksquare$

DEFINITION 17.6: The function f is **Riemann integrable** (or simply **integrable**) on $[a, b]$, with integral I, if $L(f) = U(f) = I$. If this is the case, we write $\int_a^b f(x)\,dx = I$.

Notice that there does not seem to be a limit process involved in the definition of the integral! (At least the limit process is well hidden.)

EXAMPLES 17.1: 1. If $f(x) = C$ for all x, then $L(f, P) = U(f, P) = C(b-a)$ for any partition P. Thus any constant function is integrable, with integral $C(b-a)$.

2. Let $f(x) = 0$ for $x \neq 0$ and $f(0) = 1$. We will show that $\int_{-1}^1 f(x)\,dx = 0$. Since $m_k = 0$ for any interval, we have $L(f, P) = 0$ for any partition, and so $L(f) = 0$. We may assume each partition has 0 as an element (since adding a point to a partition produces a refinement of it). Now if $0 \notin [x_{k-1}, x_k]$, we have $M_k = 0$. If $x_j = 0$, then $M_j = M_{j-1} = 1$ and $U(f, P) = x_{j+1} - x_{j-1}$. Now $x_{j+1} - x_{j-1} > 0$, and it can be made as small as we wish by refining P. Thus $U(f) = 0 = L(f)$.

3. Let $D(x)$ be the Dirichlet function over the interval $[0, 1]$. Then, for any partition P, $U(D, P) = 1$ and $L(D, P) = 0$. Thus $U(D) = 1$ and $L(D) = 0$, and D is not integrable over $[0, 1]$.

4. We verify that $\int_0^1 x\,dx = \frac{1}{2}$. Let $P_n = \{0, 1/n, 2/n, \ldots, 1\}$ and examine upper and lower sums based on P_n. Since $f(x) = x$ is increasing, we have $m_k = (k-1)/n$ and $M_k = k/n$ for all k. Then

$$\begin{aligned} & L(f, P_n) \\ = \quad & \sum_{k=1}^n \left(\frac{k-1}{n}\right)\left(\frac{1}{n}\right) \\ = \quad & \frac{1}{n^2}\sum_{k=1}^n (k-1) \\ = \quad & \frac{1}{2} - \frac{1}{n^2}. \end{aligned}$$

Similarly, $U(f, P_n) = 1/2 + 1/n^2$. We know, then, that

$$1/2 - 1/n^2 \leq L(f) \leq U(f) \leq 1/2 + 1/n^2$$

for any n. It follows that $L(f) = U(f) = 1/2$.

The argument of Example 4 can be generalized, giving us the following result.

THEOREM 17.7: *Suppose there exists a collection of partitions* $\{P_n\}$ *with* $\inf U(f, P_n) = \sup L(f, P_n) = I$. *Then* $\int_a^b f(x)\,dx = I$. *If, furthermore,* $U(f, P_{n+1}) \le U(f, P_n)$ [*or* $L(f, P_{n+1}) \ge L(f, P_n)$] *for all* n, *then* $I = \lim_{n\to\infty} U(f, P_n)$ [*or* $I = \lim_{n\to\infty} L(f, P_n)$]. ∎

Theorem 17.7 is our first real indication that the integral might be computed using a limit (in certain circumstances, the limit of a sequence). Keep in mind that we are usually interested primarily in whether an integral exists and not so much in its value if it does. The Cauchy criterion has always been a powerful tool for obtaining information of this type. Here is a version of it for integrals.

THEOREM 17.8: $f : [a, b] \to \mathbf{R}$ *is integrable if and only if, for any* $\varepsilon > 0$, *there is a partition* P *so that* $U(f, P) - L(f, P) < \varepsilon$.

PROOF: This follows from Exercise 5.1.11 and Lemma 17.3. ∎

EXERCISES 17.1

1. Show in detail that $\int_0^1 x^2\,dx = \frac{1}{3}$.

2. Convince yourself that the containment in the definition of "refinement" goes the right way.

3. We have assumed that the functions we are dealing with are bounded. Show that a function that is Riemann integrable *must* be bounded.

4. If f is integrable on $[a, b]$ and $a < c < b$, show that f is integrable on $[a, c]$ and on $[c, b]$ and that $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$.

5. (a) If $f : [a, b] \to \mathbf{R}$ is integrable, show that the restriction of f to any interval $[c, d] \subseteq [a, b]$ is integrable.

   (b) Let $f : [0, 1] \to \mathbf{R}$. If f is integrable on $[\varepsilon, 1]$ for every $0 < \varepsilon < 1$, is it necessarily true that f is integrable on [0, 1]?

   (c) Can a hypothesis be added in (b) to make the conclusion true?

   (d) If $f : [a, b] \to \mathbf{R}$ is such that the restriction of f to any interval $[c, d]$ that is properly contained in $[a, b]$ is integrable, is f necessarily integrable? Can a condition be added to make it so?

6. Draw a picture to illustrate the argument in Example 17.1.2.

7. What theorem about the real line is used in Example 17.1.3?

8. Show that the function

$$f(x) = \begin{cases} \sin(1/x) & x > 0 \\ 0 & x = 0 \end{cases}$$

is integrable on $[0,1]$.

9. The average value of a collection of numbers $a_1, a_2, \ldots, a_n$ is that number, a, so that if each term in $a_1 + a_2 + \cdots + a_n$ is replaced by a, the sum remains the same. Explain why it is reasonable in this sense to define the average value of a function $f : [a,b] \to \mathbf{R}$ to be $\frac{1}{b-a}\int_a^b f(x)\,dx$.

17.2 OSCILLATIONS

The various conditions for integrability we have discussed each have their own conceptual advantages. The definition itself suggests the familiar integration process but requires evaluation of suprema, infima, or limits. Theorem 17.8 doesn't seem to involve any limit process, but it's hard to see it as "integration." Since $U(f,P) - L(f,P) = \sum(M_k - m_k)\Delta x_k$, we might simply ask the latter sum to be small. This gives us only one sum to consider. The difference $M_k - m_k$ is called the **oscillation** of f over $[x_{k-1}, x_k]$, denoted $\operatorname{osc}(f, [x_{k-1}, x_k])$. With this terminology, the condition in Theorem 17.8 becomes:

COROLLARY 17.9: *The function $f : [a,b] \to \mathbf{R}$ is integrable if and only if, for any $\varepsilon > 0$, there is a partition P so that*

$$\sum \operatorname{osc}(f, [x_{k-1}, x_k])\Delta x_k < \varepsilon. \blacksquare$$

EXAMPLES 17.2: 1. Here we show again that the function $f(x) = x$ is integrable on $[0,1]$. For any interval $[x_{k-1}, x_k]$, we have $\operatorname{osc}(f, [x_{k-1}, x_k]) = x_k - x_{k-1}$. Thus, for any partition P,

$$\begin{aligned} & \sum \operatorname{osc}(f, [x_{k-1}, x_k])\Delta x_k \\ = & \sum (x_k - x_{k-1})\Delta x_k \\ \le & \left(\max_{1 \le k \le n}\{x_k - x_{k-1}\}\right) \sum \Delta x_k \\ = & \max_{1 \le k \le n}\{x_k - x_{k-1}\} \end{aligned}$$

since $\sum \Delta x_k = 1$. If $\varepsilon > 0$ is given, the condition of Corollary 17.9 is satisfied by any partition with $\max_{1 \le k \le n}\{x_k - x_{k-1}\} < \varepsilon$. The quantity $\max_{1 \le k \le n}\{x_k - x_{k-1}\}$ is called the **mesh** of P, denoted $\mu(P)$.

We make a few observations about oscillations:

LEMMA 17.10: (a) *If* $f(x) - f(y) \le B$ *for* $x, y \in S$, *then* $\operatorname{osc}(f, S) \le B$.
(b) *If* f *is continuous and* S *is compact, there are points* $x, y \in S$ *so that* $\operatorname{osc}(f, S) = f(x) - f(y)$.
(c) $\operatorname{osc}(f + g, S) \le \operatorname{osc}(f, S) + \operatorname{osc}(g, S)$.
(d) *If* $c \in \mathbf{R}$, $\operatorname{osc}(cf, S) = |c|(\operatorname{osc}(f, S))$.

PROOF: We will prove (a), leaving the rest as Exercise 17.2.2. We use the technique of Exercise 4.6.13. Let $u = \sup_{x \in S}\{f(x)\}$ and $v = \inf_{x \in S}\{f(x)\}$. Let $\varepsilon > 0$ be given. There are points $x, y \in S$ so that $f(x) + \varepsilon/2 > u$ and $f(y) - \varepsilon/2 < v$. Thus for any $\varepsilon > 0$, we have

$$\operatorname{osc}(f, S) = u - v < f(x) - f(y) + \varepsilon \le B + \varepsilon.$$

By Exercise 4.6.13, $\operatorname{osc}(f, S) \le B$. ∎

THEOREM 17.11: *If* $f : [a, b] \to \mathbf{R}$ *and* $g : [a, b] \to \mathbf{R}$ *are integrable and* $c \in \mathbf{R}$, *then* $f + g$ *and* cf *are integrable and*

$$\int_a^b f(x) + g(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$

and

$$\int_a^b cf(x)\, dx = c \int_a^b f(x)\, dx$$

PROOF: We will prove that $f + g$ is integrable and leave cf and the formulas as Exercise 17.2.1 (proving such a formula is different from showing integrability). Let $\varepsilon > 0$ be given and let P be such that $\sum \operatorname{osc}(f, [x_{k-1}, x_k])\Delta x_k < \varepsilon/2$ and $\sum \operatorname{osc}(g, [x_{k-1}, x_k])\Delta x_k < \varepsilon/2$. By Lemma 17.10,

$$\begin{aligned} & \sum \operatorname{osc}(f + g, [x_{k-1}, x_k])\Delta x_k \\ \le\ & \sum \operatorname{osc}(f, [x_{k-1}, x_k])\Delta x_k + \sum \operatorname{osc}(g, [x_{k-1}, x_k])\Delta x_k \\ <\ & \varepsilon, \end{aligned}$$

and so $f + g$ is integrable. ∎

EXERCISES 17.2

1. Prove Corollary 17.9.

2. Complete the proof of Lemma 17.10.

3. Prove Theorem 17.11.

4. Suppose $\|f-g\|_\infty < \varepsilon$ on S. Show that
$$\operatorname{osc}(g,S) - 2\varepsilon < \operatorname{osc}(f,S) < \operatorname{osc}(g,S) + 2\varepsilon.$$

5. (a) If $f : [a,b] \to \mathbf{R}$ is Riemann integrable, show that the function f^2 is integrable on $[a,b]$. (Hint: Consider the oscillation of f^2. Recall that an integrable function is bounded.)

   (b) Using the identity $(f+g)^2 = f^2 + 2fg + g^2$, show that the *product* of two Riemann integrable functions is Riemann integrable.

17.3 INTEGRABILITY OF CONTINUOUS FUNCTIONS

We turn to the main theorem of the chapter. In abstract theories of integration one is not generally concerned with the value of a specific integral. The important questions are whether a function is integrable or not and whether the class of all integrable functions can be identified. We begin that process now.

THEOREM 17.12: *If $f : [a,b] \to \mathbf{R}$ is continuous, then f is Riemann integrable.*

PROOF: Since $[a,b]$ is compact, f is uniformly continuous (the intrusion of uniform continuity puts this proof "beyond the scope" of elementary calculus). Then given $\varepsilon > 0$, there is a $\delta > 0$ so that $|f(x) - f(y)| < \varepsilon$ whenever $d - c < \delta$ and $x, y \in [c,d]$. According to Lemma 17.10, this means that $\operatorname{osc}(f,[c,d]) < \varepsilon$ for any such $[c,d]$. This is just what we need. Let $\varepsilon > 0$ be given and $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon/(b-a)$. Choose a partition $P = \{x_0, \ldots, x_n\}$ with $\mu(P) < \delta$. Then

$$\begin{aligned}
& \sum \operatorname{osc}(f,[x_{k-1},x_k])\Delta x_k \\
< \; & \sum \frac{\varepsilon}{(b-a)}\Delta x_k \\
= \; & \frac{\varepsilon}{(b-a)} \sum \Delta x_k \\
= \; & \varepsilon
\end{aligned}$$

since $\sum \Delta x_k = b - a$. Thus f is integrable by Corollary 17.9. ∎

In Example 17.1.2 we saw that a discontinuous function can be integrable but that something as wildly discontinuous as the Dirichlet function is not. We will find how badly discontinuous a function can be and still be

integrable in Chapter 21, and for now will examine only one more result of this type. A monotone function can have many discontinuities (just how many it can have will be explored in Chapter 19). Nevertheless ...

THEOREM 17.13: *If $f : [a, b] \to \mathbf{R}$ is monotone, then f is Riemann integrable.*

PROOF: We will assume f is increasing. Let P be any partition of $[a, b]$. For each interval $[x_{k-1}, x_k]$, we have $M_k = f(x_k)$ and $m_k = f(x_{k-1})$. Thus

$$\begin{aligned} & \sum \operatorname{osc}(f, [x_{k-1}, x_k])\Delta x_k \\ = & \sum (f(x_k) - f(x_{k-1}))\Delta x_k \\ \leq & \ \mu(P) \sum (f(x_k) - f(x_{k-1})) \\ = & \ \mu(P)(f(b) - f(a)) \end{aligned}$$

and Corollary 17.9 will be satisfied by taking any partition P with $\mu(P) < \varepsilon/(f(b) - f(a))$. ∎

EXERCISES 17.3

1. Complete the proof of Lemma 17.3.a.

2. (a) If $f : [a, b] \to \mathbf{R}$ is continuous and $f(x) \geq 0$ for all $x \in [a, b]$, show that $\int_a^b f(x)\,dx \geq 0$.
   (b) If $f : [a, b] \to \mathbf{R}$ is continuous, $f(x) \geq 0$ for all $x \in [a, b]$, and there is at least one number c for which $f(c) > 0$, show that $\int_a^b f(x)\,dx > 0$.
   (c) Show that (a) remains true if it is assumed only that f is integrable.
   (d) Show that (b) does *not* remain true if it is assumed only that f is integrable.

3. (a) Prove the **Mean Value Theorem for Integrals**: If $f : [a, b] \to \mathbf{R}$ is continuous, there is a number $c \in [a, b]$ so that
   $$\int_a^b f(x)\,dx = f(c)(b - a).$$
   (b) Describe this result geometrically.

4. (a) Draw a picture and describe the geometric significance of the expression
   $$\frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right).$$

(b) If f is continuous on $[a, b]$, show that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right).$$

(c) Give an example of a nonintegrable function where the limit in (b) exists. (This is why we can't use this as the definition of the integral.)

5. We say that $f : [a, b] \to \mathbf{R}$ is a **step function** if $[a, b]$ can be decomposed into finitely many subintervals (some of which might consist of only one point) on each of which f is constant.

   (a) Show that a step function has a finite range. (Using this observation, we can define a step function for a domain that is bounded but not compact.)

   (b) Show that a step function whose domain is a closed, bounded interval is Riemann integrable.

   (c) If $f : [a, b] \to \mathbf{R}$ is Riemann integrable and $\varepsilon > 0$ is given, show that there is a step function g such that $\int_a^b |f(x) - g(x)|\,dx < \varepsilon$.

   (d) If $f : [a, b] \to \mathbf{R}$ is a step function and $\varepsilon > 0$ is given, show that there is a *continuous* function g such that $\int_a^b |f(x) - g(x)|\,dx < \varepsilon$.

   (e) If $f : [a, b] \to \mathbf{R}$ is Riemann integrable and $\varepsilon > 0$ is given, show that there is a continuous function g such that $\int_a^b |f(x) - g(x)|\,dx < \varepsilon$.

6. A function $f : [a, b] \to \mathbf{R}$ is **piecewise linear** if f is continuous and $[a, b]$ can be decomposed into finitely many intervals on each of which f is a straight line.

   (a) Draw a graph of a piecewise linear function.

   (b) If $f : [a, b] \to \mathbf{R}$ is continuous and $\varepsilon > 0$ is given, show that there is a piecewise linear function g such that $\|f - g\|_\infty < \varepsilon$.

   (c) If $f : [a, b] \to \mathbf{R}$ is continuous and $\varepsilon > 0$ is given, show that there is a piecewise linear function g such that $\int_a^b |f(x) - g(x)|\,dx < \varepsilon$.

   (d) If $f : [a, b] \to \mathbf{R}$ is Riemann integrable and $\varepsilon > 0$ is given, show there is a piecewise linear function g such that $\int_a^b |f(x) - g(x)|\,dx < \varepsilon$.

7. Suppose $f : [a, b] \to \mathbf{R}$ is continuous and $f(x) \geq 0$ for all x. Show that

$$\lim_{p\to\infty} \left(\int_a^b (f(x))^p\,dx \right)^{1/p} = \max\{f(x) : x \in [a, b]\}.$$

8. Show that a *decreasing* function is Riemann integrable.

9. Suppose $f : [a, b] \to \mathbf{R}$ is such that $[a, b]$ can be decomposed into a finite collection of intervals, on each of which f is monotone. Show that f is integrable.

17.4 THE FUNDAMENTAL THEOREMS

It is not our purpose to review the computational techniques of calculus. On the other hand, the results called the "Fundamental theorems" should not be left unmentioned (and their proofs make use of the Big Theorem). Other computational devices, so important in elementary calculus, aren't as close to the structure of the real numbers, and we won't discuss them here.

THEOREM 17.14: (The Fundamental Theorem of Calculus) *If the function $f : [a, b] \to \mathbf{R}$ is continuous and $F : [a, b] \to \mathbf{R}$ is such that $F'(x) = f(x)$ for $x \in (a, b)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.*

PROOF: Since f is continuous, it is integrable, say with integral I. Let $\varepsilon > 0$ be given. We will show that $|F(b) - F(a) - I| < \varepsilon$. Let $P = \{x_0, \ldots, x_n\}$ be a partition with $U(f, P) - L(f, P) < \varepsilon$. We write $F(b) - F(a)$ in the following way (remember $a = x_0$ and $b = x_n$):

$$\begin{aligned} & F(b) - F(a) \\ = \quad & F(x_n) - F(x_{n-1}) + F(x_{n-1}) - \cdots - F(x_1) + F(x_1) - F(x_0). \end{aligned}$$

By the Mean Value theorem, there is, for each k, a number $\xi_k \in [x_{k-1}, x_k]$ so that $F(x_k) - F(x_{k-1}) = F'(\xi_k)(x_k - x_{k-1}) = f(\xi_k)(x_k - x_{k-1})$. Thus $F(b) - F(a) = \sum f(\xi_k)\Delta x_k$. Now both $L(f, P) \le \sum f(\xi_k)\Delta x_k \le U(f, P)$ and $L(f, P) \le I \le U(f, P)$. It follows that $|F(b) - F(a) - I| = |\sum f(\xi_k)\Delta x_k - I| < \varepsilon$ (by Theorem 4.19). Since this is true for any $\varepsilon > 0$, we have $F(b) - F(a) = I$. ∎

The Fundamental theorem tells us, in a sense, what happens if we "integrate a derivative." The next theorem, called the **Second Fundamental Theorem** (sometimes the billing is reversed), tells us what happens if we "differentiate an integral." First we need a lemma, itself a result of some importance.

LEMMA 17.15: *If $f : [a, b] \to \mathbf{R}$ is integrable, then $|f|$ is integrable and*

$$\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx.$$

PROOF: To establish that $|f|$ is integrable we must first estimate its oscillation. By the Triangle inequality, for any $x, y \in [x_{k-1}, x_k]$,

$$||f(x)| - |f(y)|| \leq |f(x) - f(y)| \leq \operatorname{osc}(f, [x_{k-1}, x_k]).$$

It follows by Lemma 17.10 that $\operatorname{osc}(|f|, [x_{k-1}, x_k]) \leq \operatorname{osc}(f, [x_{k-1}, x_k])$. Let $\varepsilon > 0$ be given and P be a partition with $\sum \operatorname{osc}(f, [x_{k-1}, x_k])\Delta x_k < \varepsilon$. Then

$$\begin{aligned} & \sum \operatorname{osc}(|f|, [x_{k-1}, x_k])\Delta x_k \\ < \; & \sum \operatorname{osc}(f, [x_{k-1}, x_k])\Delta x_k \\ < \; & \varepsilon \end{aligned}$$

and $|f|$ is integrable by Corollary 17.9. By the Triangle inequality again, $|U(f,P)| \leq U(|f|, P)$ and $|L(f,P)| \leq L(|f|, P)$ for any partition P. The rest of the lemma follows. ∎

THEOREM 17.16: (The Second Fundamental Theorem) *Suppose that $f : [a,b] \to \mathbf{R}$ is integrable and $F : [a,b] \to \mathbf{R}$ is defined by*

$$F(x) = \int_a^x f(t)\, dt.$$

Then

(a) *$F(x)$ is uniformly continuous.*

(b) *If f is continuous at a point c, then F is differentiable at c and $F'(c) = f(c)$.*

PROOF: (a) We have defined the Riemann integral only for bounded functions, though it might not be clear that this is a necessary restriction. You showed in Exercise 17.1.3 that an integrable function must be bounded. Suppose $|f(x)| < B$ for all $x \in [a,b]$, then for $x, y \in [a,b]$,

$$\begin{aligned} & |F(y) - F(x)| \\ = \; & \left| \int_a^y f(t)\, dt - \int_a^x f(t)\, dt \right| \\ = \; & \left| \int_x^y f(t)\, dt \right| \quad \text{(by Exercise 17.1.4)} \\ \leq \; & \int_x^y |f(t)|\, dt \quad \text{(by Lemma 17.15)} \\ \leq \; & B(|y - x|). \end{aligned}$$

It follows that F is uniformly continuous.

(b) Now suppose f is continuous at c. To show that $F'(c) = f(c)$, we

must examine $F(x) - F(c) - f(c)(x-c)$. Inserting the definition of F, using Exercise 17.1.4, and noting that $f(c)(x-c) = \int_c^x f(c)\,dt$, we have

$$\begin{aligned} & F(x) - F(c) - f(c)(x-c) \\ = \; & \int_a^x f(t)\,dt - \int_a^c f(t)\,dt - \int_c^x f(c)\,dt \\ = \; & \int_c^x f(t) - f(c)\,dt \end{aligned}$$

Let $\eta(s) = \sup|f(c+s) - f(c)|$. Then $\eta(s) \to 0$ as $s \to c$ (since f is continuous at c). Then $\left|\int_c^x f(t) - f(c)\,dt\right| \le \eta(x-c)(x-c) = o(x-c)$, and so $F'(c) = f(c)$. ∎

The proof of the (first) Fundamental theorem is quite reminiscent of the approach to integration that is familiar from calculus. It seems that the proof would have been a bit easier if we could have dealt with the points ξ_k directly, without having to make the estimate in the second from last line. This approach to the problem is what is usually called "Riemann integration."

DEFINITION 17.17: (a) A collection of points $\Xi = \{\xi_1, \ldots, \xi_n\}$ is called a set of **intermediate points** to the partition $P = \{x_0, \ldots, x_n\}$ if $x_{k-1} \le \xi_k \le x_k$ for $k = 1, \ldots, n$.

(b) The **Riemann sum** for $f : [a, b] \to \mathbf{R}$ over the partition P with intermediate points Ξ is $R(f, P, \Xi) = \sum_{k=1}^n f(\xi_k)\Delta x_k$.

THEOREM 17.18: $\int_a^b f(x)\,dx = I$ *if and only if given* $\varepsilon > 0$ *there is a partition* P *so that if* P' *is any refinement of* P *and* Ξ *is any collection of intermediate points of* P'*, then* $|R(f, P', \Xi) - I| < \varepsilon$.

PROOF: The "only if" part has been done in the proof of the Fundamental theorem. Suppose the condition of the theorem holds and let P be such that $|R(f, P', \Xi) - I| < \varepsilon/3$ for any refinement of P. Then $U(f, P') - I \le \varepsilon/3$ and $I - L(f, P') \le \varepsilon/3$, and so $U(f, P') - L(f, P') < \varepsilon$. ∎

EXERCISES 17.4

1. (a) Show that "the rest of the lemma follows" as claimed in the proof of Lemma 17.15.

   (b) If f is integrable and g is continuous, show that $g \circ f$ is integrable.

   (c) Show that Lemma 17.15 follows from this.

2. Let F be defined as in the Second Fundamental theorem:

$$F(x) = \int_a^x f(t)\,dt.$$

(a) If f is positive, show that F is increasing.

(b) If f is increasing, show that F is convex (see Exercise 16.7.6).

(c) The observation in (b) is often used to construct convex functions with specified properties. For instance, show that if $f(x) \to 0$ as $x \to a^+$, then $F(x) = o(x)$ as $x \to a^+$.

(d) State and prove a condition on f similar to the one in (c) that will guarantee that $F(x) = o(x^2)$.

3. (a) Show that $f(x) = \int_0^x |t|\,dt$ is differentiable at $x = 0$ but does not have a second derivative at $x = 0$.

(b) Construct a function g such that $g'(a), g''(a), \ldots, g^{(n)}(a)$ all exist for some point a, but $g^{(n+1)}(a)$ does not exist.

4. If f and g are continuous functions such that $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$, show that there must be a number $c \in (a, b)$ with $f(c) = g(c)$.

5. Show how the "only if" part of Theorem 17.18 "...has been done in the proof of the Fundamental theorem."

6. Here is an outline of a proof of the Mean Value theorem. Suppose that $f(x)$ and $g(x)$ are functions satisfying the hypotheses of the theorem and that $g(a) = f(a)$. Let $S = [f(b) - f(a)]/(b - a)$.

(a) Suppose that $g'(x) > S$ for all $x \in (a, b)$. Use the Second Fundamental theorem to show that $g(b) > f(b)$.

(b) Suppose that $g'(x) < S$ for all $x \in (a, b)$. Use the Second Fundamental theorem to show that $g(b) < f(b)$.

(c) Show that unless $f'(x) = S$ for all x, it must be the case that there is a point $\alpha \in (a, b)$ with $f'(\alpha) > S$ and a point $\beta \in (a, b)$ with $f'(\beta) < S$.

(d) Use Exercise 12.6.2 to show that there is a $c \in (a, b)$ with $f'(c) = 0$.

(e) Discuss whether this proof is valid.

17.5 RIEMANN-STIELTJES INTEGRATION

Integration as seen in calculus is mainly about areas under curves. We realize as we go along that this is not the essence of integration (though it is a very useful application). If we aren't tied to this geometric problem, we might ask whether the process we have described is the only one that

can reasonably be called "integration." You have gathered from the title of the section that it is not. The Riemann integral is an averaging process of sorts, in which values of a function are weighted with the lengths of the subintervals in a partition and added. We will alter the process by using a second function to determine the weights associated with the subintervals.

DEFINITION 17.19: (a) If $f : [a,b] \to \mathbf{R}$ and $g : [a,b] \to \mathbf{R}$ are bounded, P is a partition of $[a,b]$, and Ξ is a collection of intermediate points of P, the **Riemann-Stieltjes sum** of f with respect to g, P, and Ξ is

$$R(f,g,P,\Xi) = \sum_{k=1}^{n} f(\xi_k)(g(x_k) - g(x_{k-1})).$$

(b) f is **Riemann-Stieltjes integrable with respect to g** with integral I if, given $\varepsilon > 0$, there is a partition P so that $|R(f,g,P',\Xi) - I| < \varepsilon$ whenever P' is a refinement of P and for any collection of intermediate points. If this is so, we write

$$\int_a^b f\,dg = I.$$

The function g is called the **integrator** in this expression.

We have adopted a definition of the Riemann-Stieltjes integral based on "intermediate points" because it will make an important theorem easier to prove later on. We can still consider "upper and lower sums," though. The proof of the following theorem will be left as Exercise 17.5.2. We write $\Delta g_k = g(x_k) - g(x_{k-1})$.

THEOREM 17.20: *If g is increasing, f is Riemann-Stieltjes integrable with respect to g if and only if, given $\varepsilon > 0$, there is a partition P so that*

$$U(f,g,P) - L(f,g,P) = \sum M_k \Delta g_k - \sum m_k \Delta g_k < \varepsilon,$$

where U, L, M_k, and m_k are defined as before. ■

We have an analogue of Corollary 17.9:

COROLLARY 17.21: *If f is bounded and g is increasing, then f is Riemann-Stieltjes integrable with respect to g if and only if, for $\varepsilon > 0$, there is a partition $P = \{x_0, x_1, \ldots, x_n\}$ so that*

$$\sum \operatorname{osc}(f, [x_{k-1}, x_k])\Delta g_k < \varepsilon.$$ ■

EXAMPLES 17.5: 1. If $g(x) = x$, the Riemann-Stieltjes integral $\int_a^b f\,dg$ is the same as the Riemann integral $\int_a^b f(x)\,dx$.

2. Suppose $f : [-1, 1] \to \mathbf{R}$ is continuous at 0. Let $g(x) = 0$ if $x \leq 0$ and $g(x) = 1$ if $x > 0$. Then $g(x_k) - g(x_{k-1}) = 0$ unless $0 \in [x_{k-1}, x_k)$, and if this is so, $g(x_k) - g(x_{k-1}) = 1$. Then $U(f, g, P) = \sup_{x \in [x_{k-1}, x_k)} f(x)$ and $L(f, g, P) = \inf_{x \in [x_{k-1}, x_k)} f(x)$. Since f is continuous at 0, both these values can be made close to $f(0)$ by making $x_k - x_{k-1}$ small. It follows that $\int_{-1}^{1} f\,dg = f(0)$.

3. Let $g(x)$ be as in Example 2. We will show that g is not Riemann-Stieltjes integrable with respect to itself.[1] Let P be a partition that contains 0. Suppose $x_k = 0$. Note that $\operatorname{osc}(g, [x_{j-1}, x_j]) = 0$ and $\Delta g_j = 0$, except for $j = k + 1$, and $\operatorname{osc}(g, [x_k, x_k + 1]) = \Delta g_{k+1} = 1$. Thus $\sum \operatorname{osc}(g, [x_{k-1}, x_k])\Delta g_k = 1$, and the result follows from Corollary 17.21.

Since the condition of Corollary 17.21 is much easier to check than the definition, we will prove most of the theorems that follow for increasing integrators. In the end we will see that this restriction does not hamper us much.

EXERCISES 17.5

1. Let $f(x) = x$ and $g(x) = \sin(x)$. Evaluate $\int_{-\pi/2}^{\pi/2} f\,dg$.

2. Prove Theorem 17.20.

3. (a) If $f : [a, b] \to \mathbf{R}$ is continuous, $c \in (a, b)$, and g is the function

$$g(x) = \begin{cases} 0 & x \leq c \\ 1 & x > c, \end{cases}$$

evaluate $\int_a^b f\,dg$.

(b) If $f : [a, b] \to \mathbf{R}$ is continuous and $x_1, \ldots, x_n \in [a, b]$, construct a Riemann-Stieltjes integral whose value is $f(x_1) + \cdots + f(x_n)$.

(c) Suppose (x_n) is an increasing sequence in $[a, b]$ with $\lim x_n = b$ and $\sum \alpha_n$ is a convergent, positive series. Construct a Riemann-Stieltjes integral whose value is $\sum \alpha_n f(x_n)$.

(d) Suppose (x_n) is a sequence contained in $[a, b]$. Is there necessarily a Riemann-Stieltjes integral whose value is $\sum f(x_n)$?

---

[1] That this can happen for such a simple function has led many people to consider alternative definition of the Riemann-Stieltjes integral

4. If f is Riemann-Stieltjes integrable with respect to the differentiable function g, show that

$$\int_a^b f\,dg = \int_a^b f(x)g'(x)\,dx.$$

(Note that the last expression is a *Riemann* integral.)

17.6 RIEMANN-STIELTJES INTEGRABLE FUNCTIONS

THEOREM 17.22: *If $f : [a,b] \to \mathbf{R}$ is continuous and $g : [a,b] \to \mathbf{R}$ is increasing, then f is Riemann-Stieltjes integrable with respect to g.*

PROOF: Since f is uniformly continuous, $\operatorname{osc}(f,[x,y])$ can be made small uniformly (that is, everywhere in the interval at once) by making $y-x$ small. Let $\varepsilon > 0$ be given and let $\delta > 0$ be such that $\operatorname{osc}(f,[x,y]) < \varepsilon/(g(b)-g(a))$ whenever $y-x<\delta$. If P is any partition with $\mu(P) < \delta$, we have

$$\begin{aligned} & \sum \operatorname{osc}(f,[x_{k-1},x_k])\Delta g_k \\ < \ & \frac{\varepsilon}{g(b)-g(a)} \sum \Delta g_k \\ = \ & \varepsilon \end{aligned}$$

since $\sum \Delta g_k = g(b) - g(a)$. ∎

All of our arguments can be adjusted to hold for decreasing integrators. The following theorem, whose proof is Exercise 17.6.1, can be combined with this observation to open important avenues in the theory.

THEOREM 17.23: *If f is integrable with respect to both g and h, then f is integrable with respect to $g+h$ and*

$$\int_a^b f\,d(g+h) = \int_a^b f\,dg + \int_a^b f\,dh. \ ∎$$

A continuous function is thus integrable with respect to any function that can be written as a sum of an increasing function and a decreasing function. Such functions are said to be of **bounded variation**.

COROLLARY 17.24: *If $f : [a,b] \to \mathbf{R}$ is continuous and $g : [a,b] \to \mathbf{R}$ is of bounded variation, then f is integrable with respect to g.* ∎

You will examine functions of bounded variation in Exercise 17.7.6.

EXERCISES 17.6

1. Prove Theorem 17.23.

2. (a) Suppose g is increasing and f is bounded. If f is continuous at all points where g is discontinuous, show that f is Riemann-Stieltjes integrable with respect to g.

   (b) Let $f(x) = 0$ for $x \leq 0$ and $f(x) = 1$ for $x > 0$ and let $g(x) = 0$ for $x < 0$ and $g(x) = 1$ for $x \geq 0$. Show that f is integrable with respect to g but that neither f nor g is integrable with respect to itself.

   (c) Show that h is integrable with respect to f [in part (b)] if and only if h is continuous *from the right* at 0.

   (d) Show that h is integrable with respect to g [in part (b)] if and only if h is continuous *from the left* at 0.

   (e) If g is increasing, show that f is integrable with respect to g if and only if there are no points where f and g have a common one-sided discontinuity (that is, where they are discontinuous *from the same side*).

17.7 INTEGRATION BY PARTS

Corollary 17.24 goes a long way toward settling one of the most important questions of Riemann-Stieltjes integration, but it doesn't tell the whole story. There are discontinuous functions and functions that are not of bounded variation that can be integrated. Integration by parts, a familiar subject from calculus, opens more possibilities. This may have seemed a purely symbolic exercise in calculus, but it is considerably deeper and speaks to the subtle interplay of integrator and integrand in a Riemann-Stieltjes integral. The proof of this theorem is technical (it might be a good idea to draw a picture as you go along) but the formula in the conclusion of the theorem should be familiar.

THEOREM 17.25: *If f is integrable with respect to g, then g is integrable with respect to f and*

$$\int_a^b f\,dg = f(b)g(b) - f(a)g(a) - \int_a^b g\,df.$$

PROOF: Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$ such that if P' is any refinement of P and $\Xi = \{\xi_1, \ldots, \xi_n\}$ is a collection of intermediate points, $|R(f, g, P', \Xi) - I| < \varepsilon$. Let $P' = \{y_0, \ldots, y_{2n}\} = P \cup \Xi$. Then P' is also a partition of $[a, b]$, with $y_{2k} = x_k$ and $y_{2k-1} = \xi_k$, and P' is a refinement of P. Consider the sum $\sum_{k=1}^{n} g(\xi_k)\Delta f_k$. By adding and

subtracting the terms $f(y_0)g(y_0), f(y_2)g(y_2), \ldots$, rearranging, and noting that $a = y_0$ and $b = y_{2n}$, we see that

$$\begin{aligned}
&\sum_{k=1}^{n} g(\xi_k)\Delta f_k \\
= \quad & f(b)g(b) - f(a)g(a) - \sum_{k=1}^{n} f(\zeta_k)(g(y_k) - g(y_{k-1})) \\
= \quad & f(b)g(b) - f(a)g(a) - R(f, g, P', Z),
\end{aligned}$$

where $Z = \{\zeta_1, \ldots, \zeta_{2n}\}$ and ζ_k is always one of the original partition points x_k. The last sum is a Riemann-Stieltjes sum for f with respect to g over a refinement of P, and so is within ε of $\int_a^b f\,dg$, and the result follows. ∎

Combining Corollary 17.24 and Theorem 17.25, we have:

COROLLARY 17.26: *If f is of bounded variation on $[a, b]$ and g is continuous, then f is Riemann-Stieltjes integrable on $[a, b]$ with respect to g.* ∎

Theorem 17.25 essentially doubles the collection of Riemann-Stieltjes integrals, but we have only scratched the surface of this subject. A detailed study would fill another book.

EXERCISES 17.7

1. Draw a picture to describe the construction of the partition P' in the proof of Theorem 17.25.

2. Write out the manipulations of the sum $\sum_{k=1}^{n} g(\xi_k)\Delta f_k$ described in the proof of Theorem 17.25.

3. (a) Let $g(x) = \sin x$. Evaluate $\int_0^{\pi/2} x\,dg$.

   (b) Find a general formula for simplifying an integral $\int_a^b x\,dg$. What properties must the function g have to make your formula valid?

4. If a mass m is placed on the x-axis at the point $x > 0$, its **moment** about the origin is mx. (You learned about moments as a child: A seesaw will balance if the moments of the masses on either end are the same.) If several masses, $m_1, m_2, \ldots, m_n$ are placed at $x_1, x_2, \ldots, x_n$, the moment of the system is $\sum_{k=1}^{n} m_k x_k$. Suppose we have a wire of variable density, placed along the x-axis between $x = a \geq 0$ and

$x = b$, and a function $g(x) =$ [the mass of the wire to the left of x]. Explain why[2] the moment of the wire is $\int_a^b x\,dg$.

5. If a probability experiment yields numerical results $r_1, r_2, \ldots, r_n$, with associated probabilities $p_1, p_2, \ldots, p_n$ (that is, the probability you get r_k is p_k), the **expected value** of the experiment is $\sum_{k=1}^{n} p_k r_k$. Now suppose that the possible results of the experiment include all numbers between a and b. Then we would have, instead of a finite collection of probabilities, a **cumulative distribution function** $p(x)$, where $p(x) =$ [the probability that the result is $\leq x$].

   (a) Explain why $p(x) = 0$ if $x < a$ and $p(x) = 1$ if $x \geq b$.

   (b) Explain why $p(x)$ is increasing.

   (c) Show that, if the probability of getting the individual result r is 0, then $p(x)$ is continuous at r.

   (d) Explain why $p(x)$ is "right-continuous," that is, $\lim_{x \to r^+} p(x) = p(r)$, but if the probability of getting r is not 0, then $p(x)$ is *not* left-continuous at r.

   (e) Describe an experiment where the probability of some specific number is not 0 (in some contexts such a number is called an **atom**).

   (f) Show that an experiment like this can have only countably many atoms. (See Exercise 13.2.4.)

   (g) Explain why the expected value of such an experiment is $\int_a^b x\,dp$.

6. In this project we describe the functions of bounded variation in a different way. Consider functions defined on an interval $[a, b]$. A collection of closed intervals $\{[x_n, y_n]\}$, each contained in $[a, b]$, is said to be **nonoverlapping** if the open intervals $\{(x_n, y_n)\}$ are disjoint (the closed intervals may intersect only at endpoints if at all). The subintervals in a partition, for instance, are nonoverlapping, but a collection of intervals need not come from a partition, and need not be finite, to be nonoverlapping. We say f is of **bounded variation** on $[a, b]$ if $\sup \sum |f(b_n) - f(a_n)|$ is finite, the supremum taken over all nonoverlapping collections of subintervals of $[a, b]$. This supremum is called the **variation** of f over $[a, b]$, denoted $V(f, a, b)$.

   (a) Show that a bounded, monotone function is of bounded variation over any interval and compute its variation.

---

[2] In the next two exercises, you are asked to "explain" some things. You should convince yourself that the statements are true with whatever degree of precision you wish, but *proofs* are beyond the scope of the book

(b) Suppose f is such that $[a, b]$ can be written as a finite collection of intervals on each of which f is monotone. Show that f is of bounded variation and describe how to find the variation of f.

(c) Show that the function given by $f(x) = x\sin(1/x)$ if $x \neq 0$ and $f(0) = 0$ is not of bounded variation, but $g(x) = x^2 \sin(1/x)$ if $x \neq 0$ and $g(0) = 0$ is of bounded variation.

(d) Show that the sum of two functions of bounded variation is of bounded variation and that any constant multiple of a function of bounded variation is of bounded variation.

(e) Show that it is possible for a collection of nonoverlapping intervals (contained in a bounded interval) to be infinite. Show that f is of bounded variation if and only if the supremum of the sums in the definition is finite when taken only over finite collections of intervals.

(f) Show that f is of bounded variation if and only if each sum used in the definition has a finite value, whether the sum has finitely or infinitely many terms.

(g) The function $V_P(f, a, b) = \sup \sum (f(b_n) - f(a_n))$ is called the **positive variation** of f, and $V_N(f, a, b) = \inf \sum (f(b_n) - f(a_n))$ is called the **negative variation** of f (the supremum and infimum being taken over all nonoverlapping collections). Show that $V(f, a, b) = V_P(f, a, b) - V_N(f, a, b)$.

(h) If $x \in (a, b]$, we define the **variation function**, the **positive variation function**, and the **negative variation function** by $v(f, x) = V(f, a, x)$, $v_P(f, x) = V_P(f, a, x)$, and $v_N(f, x) = V_N(f, a, b)$, respectively. Show that v_P is increasing and v_N is decreasing.

(i) If f is of bounded variation, show that $v(f, x) = v_P(f, x) - v_N(f, x)$ and $f(x) = v_P(f, x) + v_N(f, x)$.

(j) Show that f is of bounded variation if and only if it can be written as the sum of an increasing function and a decreasing function (the definition given here is equivalent to the one given in the chapter).

(k) Let $y \in [a, b)$. Suppose f has the property that there are numbers $c < d$ and a decreasing sequence (α_n) that converges to y, with $f(\alpha_{2n}) < c$ and $f(\alpha_{2n+1}) > d$. Draw a picture that describes this situation. Show that f is not of bounded variation. (The significance of this is discussed in Chapter 19.)

(i) Suppose f is *not* of bounded variation on $[a, b]$. Show that there is a point $c \in [a, b]$ so that f is not of bounded variation on the set $[c - \varepsilon, c + \varepsilon] \cap [a, b]$ for any $\varepsilon > 0$. (Hint: Think about the proof of the Bolzano-Weierstrass theorem.)

7. (a) Suppose f is defined on some open interval containing the point x. Show that f is continuous at x if and only if, for any $\varepsilon > 0$, there is a $\delta > 0$ so that $\operatorname{osc}(f, [\alpha, \beta]) < \varepsilon$ whenever $x \in [\alpha, \beta]$ and $\beta - \alpha < \delta$.

   (b) Suppose f is defined on $[a, b]$. Let $\{[\alpha_n, \beta_n]\}$ be a nonoverlapping collection of intervals in $[a, b]$. We say f is **absolutely continuous** on $[a, b]$ if, for any $\varepsilon > 0$, there is a $\delta > 0$ so that $\sum \operatorname{osc}(f, [\alpha_n, \beta_n]) < \varepsilon$ whenever $\sum(\beta_n - \alpha_n) < \delta$. Show that an absolutely continuous function is continuous.

   (c) Find a continuous function that is not absolutely continuous.

   (d) Show that an absolutely continuous function is of bounded variation.

   (e) If f is differentiable and $|f'(x)| \leq B$ for all $x \in [a, b]$, show that the length of the interval $f([a, b])$ is not more than $B(b - a)$. (You should begin by saying how we know that $f([a, b])$ is an interval.)

   (f) If f has a bounded derivative, show that f is absolutely continuous.

   (g) Show that the function $F(x)$ defined in the Second Fundamental theorem is *absolutely* continuous. (Some of the significance of absolute continuity is explored in Chapter 19. An absolutely continuous function is one that can be recovered by integrating its derivative.)

8. Here is another way to describe the integral of a function. Suppose $f : [a, b] \to \mathbf{R}$ is an increasing function.

   (a) Show that, if $a \leq y_1 < y_2 \leq b$, then $\{x : y_1 < f(x) \leq y_2\}$ is an interval.

   (b) Suppose the range of f is $[c, d]$. Let $P = \{c = y_0, y_1, \ldots, y_n = d\}$ be a partition of $[c, d]$, and construct the sum $\sum_{k=1}^{n} y_k L_k$, where L_k is the length of the interval $\{x : y_{k-1} < f(x) \leq y_k\}$. Draw a picture describing this.

   (c) Let $f(x) = \sqrt{x}$ and $[a, b] = [0, 1]$ (so that $[c, d]$ is also $[0, 1]$), and suppose the points of P are equally spaced. Compute the sum in (b) for $n = 4$.

   (d) Find the limit as $n \to \infty$ of the sums in (c). Compare this with $\int_0^1 \sqrt{x}\, dx$.

   (e) If f is any increasing function, show that the limit of the sums described in (b) is $\int_a^b f(x)\, dx$.

   (f) Discuss how this process would have to be modified to allow for functions that are not increasing. (You will find, eventually, that you will be thinking about the material in Section 21.4. This is the very *tiniest* beginning of what is called **Lebesgue integration**.)

Chapter 18

Interchanging Limit Processes

18.1 A RECURRING PROBLEM

The deepest theorems in calculus describe what happens when we carry out multiple limiting processes in different orders. Our naive hope is that the result of such a process should not depend on the order in which the individual limits are computed, but even in the simplest situations this is not the case. Differentiation, integration, convergence of sequences and series, and determination of continuity are all limiting processes, and so we have dealt with this unavoidable issue in Theorems 15.4, 15.5, 17.14, and 17.16. Observe that the first two of these theorems tell us, in essence, that the results of the multiple limits involved *are* independent of the order in which the limits are computed, while the latter two say something different. In this chapter, we look at more examples of this problem. We will begin with a very direct manifestation of it.

18.2 DOUBLE SEQUENCES

A **double sequence**, denoted (a_{mn}), is a real-valued function whose domain is the Cartesian product $\mathbf{N} \times \mathbf{N}$. We wish to "let m and n go to infinity," but there are many ways this can be said to occur, as illustrated in the first example.[1]

EXAMPLES 18.2: 1. Let $a_{mn} = m/(m+n)$. Notice that $\lim_{n\to\infty} a_{mn} = 0$ regardless of the value of m, while $\lim_{m\to\infty} a_{mn} = 1$ regardless of the value of n. So $\lim_{n\to\infty}(\lim_{m\to\infty} a_{mn}) = 1$ while $\lim_{m\to\infty}(\lim_{n\to\infty} a_{mn}) = 0$. There are even more possible "limits" for this double sequence. If we consider only those terms where $m = n$, we have $\lim_{m\to\infty} a_{mm} = 1/2$.

[1] There is a potential problem in this notation: Is a_{113} "a sub eleven, three," "a sub one, thirteen," or "a sub 113" of an ordinary sequence? Should we need to refer to a specific term in a double sequence, we will use commas: $a_{11,3}$ or $a_{1,13}$.

If we always pick $n = 2m$, then we have $\lim_{m\to\infty} a_{m,2m} = 1/3$. We can make this limit seem to be any number from 0 to 1 by fixing an appropriate relationship between m and n.

2. Let $b_{mn} = 1/(m^2 + n^2)$. Then $b_{mn} \to 0$ as either $m \to \infty$ or $n \to \infty$ (each regardless of the other index). Furthermore, if $\varepsilon > 0$ is given, then $b_{mn} < \varepsilon$ whenever $m^2 + n^2 > 1/\varepsilon$.

In any of the processes described in the first example, m and n can be said to "go to infinity." Which of these values is the limit of the sequence? None of them. An ordinary sequence can have only one limit. This property should probably be preserved however we define the limit of a double sequence. The first two processes in Example 1 are called **iterated limits**. Iterated limits are easy to compute, since they involve nothing more than two ordinary limits (and no need to assume a relationship between m and n). But the iterated limits of a double sequence can be different, which doesn't help us define the "limit" of a double sequence.

In Example 2, we see that we can consider the behavior of a sequence as m and n get large either independently or together. The requirement that $m^2 + n^2$ is large says that the point (m, n) is "far away from the origin" in the sense that it lies outside some large circle. It is a bit easier to require that m and n be outside a large square, and this is how we make our definition. You will show in Exercise 18.2.1 that the results are the same.

DEFINITION 18.1: The double sequence (a_{mn}) **has limit L** if, for any $\varepsilon > 0$, there is a natural number N so that $|a_{mn} - L| < \varepsilon$ whenever $m, n > N$. If this is the case, we write $\lim_{m,n\to\infty} a_{mn} = L$.

Iterated limits seem easier to compute than double limits since they are essentially just ordinary limits. Our goal should be to see how the limit of a double sequence is related to its associated iterated limits. In the first example we saw that it is possible for both iterated limits to exist while the double limit fails to exist. The existence of a double limit does not even guarantee the existence of the iterated limits, as seen in the next example.

EXAMPLES 18.2: 3. Let $c_{mn} = (-1)^{m+n}(1/m + 1/n)$. If $m, n > N$, we have

$$\begin{aligned} & |c_{mn} - 0| \\ = \; & 1/m + 1/n \\ < \; & 2/N, \end{aligned}$$

which can be made as small as we like. Thus the double limit of this sequence is 0 even though neither iterated limit exists!

Under what conditions, if any, does the convergence of the double sequence imply the existence of the iterated limits? Under what conditions can the double limit be found by evaluating the iterated limits? In the last example, the iterated limits didn't exist because it was not possible to compute the "inside" limits. The next theorem shows that this is the only way this problem can occur.

THEOREM 18.2: *Suppose that the double limit* $\lim_{m,n\to\infty} a_{mn}$ *exists and the limit* $\lim_{n\to\infty} a_{mn}$ *exists for each* m. *Then the iterated limit* $\lim_{m\to\infty}\lim_{n\to\infty} a_{mn}$ *exists and is equal to the double limit. A similar statement holds for the other iterated limit.*

PROOF: Let $a = \lim_{m,n\to\infty} a_{mn}$ and $a_m = \lim_{n\to\infty} a_{mn}$ for each m. If $\varepsilon > 0$ is given, there is a natural number N so that $|a_{mn} - a| < \varepsilon$ when $m, n > N$. This is true for all n, and the function $f(x) = |x - a|$ is continuous. Thus

$$\begin{aligned} & |a_m - a| \\ = \; & |(\lim_{n\to\infty} a_{mn}) - a| \\ = \; & \lim_{n\to\infty} |a_{mn} - a| \\ \leq \; & \varepsilon \end{aligned}$$

for $m > N$. Thus $\lim_{m\to\infty} a_m = a$, as desired. ∎

COROLLARY 18.3: *If the double limit* $\lim_{m,n\to\infty} a_{mn}$ *exists, the limit* $\lim_{n\to\infty} a_{mn}$ *exists for each* m, *and the limit* $\lim_{m\to\infty} a_{mn}$ *exists for each* n, *then the iterated limits both exist and are the same.* ∎

Getting from the iterated limits back to the double limit will not be so easy. We have seen that it is possible for both iterated limits to exist while the double limit does not. Consider $d_{mn} = (mn)/(m^2 + n^2)$. The iterated limits are both 0, while $\lim_{n\to\infty} d_{nn} = 1/2$, and so the double limit can't exist. Thus the double limit can fail to exist even when the iterated limits are the same. In this example, $\lim_{n\to\infty} d_{mn} = 0$ for any m, but the way d_{mn} approaches the limit depends on the value of m. Everything falls into place when we eliminate this dependence.

DEFINITION 18.4: Suppose $\lim_{n\to\infty} a_{mn} = a_m$ for each m. This convergence is said to be **uniform in** $\boldsymbol{m}$ if, for each $\varepsilon > 0$, there is a natural number N_ε (which does not depend on m) so that $|a_{mn} - a_m| < \varepsilon$ when $n > N_\varepsilon$. The phrase **uniform in** $\boldsymbol{n}$ is defined similarly.

Each time we have introduced an idea of "uniformity," we have first proved a theorem to the effect that the if a "double" process is nice enough, the two "single" processes are uniform in some way. This time is no different.

THEOREM 18.5: *If the double limit* $\lim_{m,n\to\infty} a_{mn}$ *exists and the limit* $\lim_{n\to\infty} a_{mn}$ *exists for each* m, *then the convergence of the latter is uniform in* m.

PROOF: Let $\varepsilon > 0$ be given. Let $\lim_{m,n\to\infty} a_{mn} = L$ and $\lim_{n\to\infty} a_{mn} = a_m$ for each m. Since the double limit exists, there is a natural number N_1 so that $|a_{mn} - L| < \varepsilon/2$ whenever $m, n > N_1$. In the proof of Theorem 18.2, we saw that $|a_{mn} - L| \le \varepsilon/2$ whenever $m > N_1$. Thus if $m, n > N_1$, we have

$$\begin{aligned} & |a_{mn} - a_m| \\ \le \;& |a_{mn} - L| + |L - a_m| \\ < \;& \varepsilon. \end{aligned}$$

We must show that this condition does not depend on m. We have shown that the condition holds for all $m > N_1$, and so it could "depend on m" only if it failed for one of $m = 1, 2, \ldots, N_1 - 1$. For each of these, there is a number N_m so that $|a_{mn} - a_m| < \varepsilon$ whenever $n > N_m$. Let $N_2 = \max\{N_m\}$. Then, for $m = 1, 2, \ldots, N_1 - 1$, we have $|a_{mn} - a_m| < \varepsilon$ whenever $n > N_2$. Finally, let $N = \max\{N_1, N_2\}$. If $n > N$, we have $|a_{mn} - a_m| < \varepsilon$ for all m. ∎

THEOREM 18.6: *If* $\lim_{n\to\infty} a_{mn}$ *exists for each* m *and* $\lim_{m\to\infty} a_{mn}$ *exists for each* n, *and if the convergence of one of these is uniform in the other index, then the double limit* $\lim_{m,n\to\infty} a_{mn}$ *exists and all three limits are equal.*

PROOF: Let $\lim_{n\to\infty} a_{mn} = x_m$ and $\lim_{m\to\infty} a_{mn} = y_n$ and suppose that the convergence of a_{mn} to x_m is uniform in m. Let $\varepsilon > 0$ be given. There is a natural number N so that $|a_{mn} - x_m| < \varepsilon/3$ whenever $n > N$ and for all m. We want to show that (x_m) converges. Now $a_{mn} \to y_n$, and so (a_{mn}) is a Cauchy sequence (with index m). If j and k are large enough (and $j, k > N$),

$$\begin{aligned} & |x_j - x_k| \\ \le \;& |x_j - a_{jn}| + |a_{jn} - a_{kn}| + |a_{kn} - x_k| \\ < \;& \varepsilon/3 + \varepsilon/3 + \varepsilon/3. \end{aligned}$$

Thus (x_m) is a Cauchy sequence and so it converges. We have shown

that the iterated limit $\lim_{m\to\infty}\lim_{n\to\infty} a_{mn}$ exists. Now we show the double limit exists. (The rest of the theorem follows from Theorem 18.2.) Let $x = \lim x_m$. There is a number M so that $|x_m - x| < \varepsilon/3$ whenever $m > M$. With N as before, we have $|a_{mn} - x| \le |a_{mn} - x_m| + |x_m - x| < \varepsilon$ whenever $m, n > \max\{M, N\}$. ∎

EXERCISES 18.2

1. (a) Show that the definition of convergence of a double sequence is equivalent to saying c_{mn} is close to L if m and n are "outside a big enough circle."

   (b) Show that the definition of convergence of a double sequence is also equivalent to saying c_{mn} is close to L if m and n are "outside a big enough rectangle," in the sense that $\lim_{m,n\to\infty} c_{mn} = L$ if and only if, for any $\varepsilon > 0$, there are numbers M and N so that $|c_{mn} - L| < \varepsilon$ whenever $m > M$ and $n > N$.

   (c) Show that the definition of convergence of a double sequence is *not* equivalent to saying c_{mn} is close to L if m and n are "outside a *wide* enough rectangle," where we might say a rectangle is "wide" if the larger of its two dimensions is big.

2. Verify that the double limit in the first example in the chapter fails to exist.

3. Let $c_{mn} = (-1)^{m+n}(1/m + 1/n)$. Show that neither iterated limit exists.

4. (a) Let $c_{mn} = (mn)/(m^2 + n^2)$. Show that both iterated limits are 0 but that the double limit doesn't exist.

   (b) Describe how $\lim_{n\to\infty} c_{mn}$ depends on m.

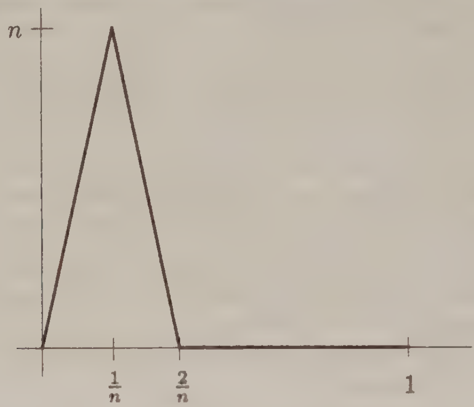
5. Show that the limit of a sum of double sequences is the sum of the limits. State and prove other simple algebraic formulas.

18.3 INTEGRALS AND SEQUENCES

The next two examples of interchange of limits have a more direct bearing on calculus and help complete the discussion of power series and Taylor series.

EXAMPLES 18.3: 1. Let $f_n(x)$ be defined by the following graph. Notice that f_n is continuous (hence integrable) for all n and the integral

of f_n is 1. Now $f(x) = \lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0, 1]$ (be sure you see that this is so). Consequently the integral of f is 0, and the "limit of the integrals" is not the same as the "integral of the limit."



We will make quick work of this problem.

THEOREM 18.7: *If (f_n) is a sequence of integrable functions with domain $[a, b]$ and $f_n \to f$ uniformly, then f is integrable on $[a, b]$ and*

$$\int_a^b \lim_{n\to\infty} f_n(x)\,dx = \int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$$

PROOF: That f is integrable follows from Exercise 17.2.4. To establish the limit formulas, we will prove the special case where $f \equiv 0$ (so that the limit on the right should be 0). Note that

$$\left| \int_a^b f_n(x)\,dx \right| \le \int_a^b |f_n(x)|\,dx \le \|f_n\|_\infty (b-a).$$

The first inequality is from Lemma 17.15. The right side goes to 0 as $\|f_n\|_\infty$ goes to 0, and the result follows. ∎

EXERCISES 18.3

1. Let $f_n(x)$ be the function described in the sketch in this section. Show that $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0, 1]$.

2. Verify the claim in the proof of Theorem 18.7 that "... f is integrable follows from Exercise 17.2.4."

3. Define a sequence of functions f_n as follows: Let $r_1, r_2, \ldots$ be an enumeration of the rational numbers. Let

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, r_2, \ldots, r_n\} \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Show that f_n is integrable on any closed, bounded interval for all n.

   (b) Show that $\lim f_n = D(x)$, the Dirichlet function, so that $\lim f_n$ is not integrable over any interval.

4. (a) Show that $\lim_{n\to\infty} \int_1^2 e^{-nx^2}\,dx = 0$.

   (b) Does the argument you used in (a) apply to $\lim_{n\to\infty} \int_0^1 e^{-nx^2}\,dx$?

   (c) Show that $\lim_{n\to\infty} \int_0^1 e^{-nx^2}\,dx = 0$.

18.4 DERIVATIVES AND SEQUENCES

Derivatives are more complicated than integrals. It may happen that the sequence (f'_n) converges uniformly while (f_n) diverges (take $f_n(x) = n$, for instance). We will see in Chapter 20 that even uniform convergence of a sequence of differentiable functions does not guarantee differentiability of the limit, much less give a means of finding its derivative. This is not particularly surprising. Functions that are uniformly close together can have derivatives that differ greatly. These difficulties are, however, cleared up in a way that is not surprising. We have to be careful to avoid the situation described above, and presently we will make the proof very easy by adding what is really an unnecessary hypothesis (continuity of the functions f'_n). This theorem is, then, only a special case, but one that includes power series. You might wish to consider how the proof would go without this added condition.

THEOREM 18.8: *Suppose f_n is differentiable on an interval (a, b) for all n and that f'_n is continuous for all n. If the sequence (f'_n) converges uniformly and there is a point $x_0 \in (a, b)$ so that $f_n(x_0)$ converges, then (f_n) converges uniformly to a differentiable function f and $\lim_{n\to\infty} f'_n = f'$.*

PROOF: By subtracting constants from each function, we may assume $f_n(x_0) = 0$ for all n [this is acceptable because $(f_n(x_0))$ converges]. By the Second Fundamental theorem, $f_n(x) = \int_{x_0}^x f'_n(x)\,dx$ for each n, and by Theorem 18.7, we have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \int_{x_0}^x f'_n(x)\,dx = \int_{x_0}^x \lim_{n\to\infty} f'_n(x)\,dx$$

If we let $g(x) = \lim_{n\to\infty} f_n'(x)$, we have, by the Second Fundamental theorem, that the function $f(x) = \int_{x_0}^{x} g(x)\,dx = \lim_{n\to\infty} f_n(x)$ is differentiable and that $f' = \lim_{n\to\infty} f_n'$ ∎

The most important uses of this result come in applying it to power series, but in order to do so, we will need to know the radius of convergence of a differentiated series. The proof of this lemma is left as Exercise 18.4.3.

LEMMA 18.9: *If $\sum a_n(x-a)^n$ has radius of convergence R, then the series of derivatives $\sum na_n(x-a)^{n-1}$ also has radius of convergence R.* ∎

Since every power series converges for at least one point (specifically, a) we can combine Theorems 15.15 and 18.9 and Lemma 18.10 to obtain:

THEOREM 18.10: *If $\sum a_n(x-a)^n$ has radius of convergence R and t is such that $|t-a| < R$, then the function given by $f(x) = \sum a_n(x-a)^n$ is differentiable at t and $f'(t) = \sum na_n(t-a)^{n-1}$.* ∎

EXAMPLES 18.4: 1. We saw in Chapter 15 that the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x. Though we suspect it strongly, this does not mean that this series is equal to e^x. We now have machinery that will let us show this. Since the radius of convergence of this series is infinite, we can find the derivative term by term for all x, and the radius of convergence of the differentiated series is also infinite (by Theorem 18.10). The differentiated series is, of course, the same as the original one. Thus the function given by $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ is such that $f'(x) = f(x)$ for all x and $f(0) = 1$. You showed in Exercise 16.5.4 that e^x is the only function satisfying these two conditions. Using this argument, we could base a study of exponential functions on the theory of series, instead of the other way around.

EXERCISES 18.4

1. Carefully state and verify that "two functions can be uniformly close together while their derivatives differ greatly."

2. (a) How is the condition of continuity of the derivatives f_n' used in the proof of Theorem 18.8?

   (b) Discuss Theorem 18.8 without this condition.

3. (a) Prove Lemma 18.9.

   (b) Show that the power series obtained by *antidifferentiating* another series term by term has the same radius of convergence as the original. (Assume that all the constants of integration are 0.)

4. Use the technique of Example 18.4.1 and Exercise 16.5.4 to show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n!)} = \cos x.$$

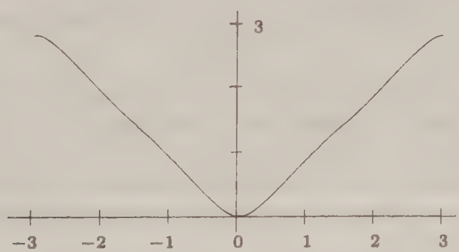
5. Prove Theorem 18.10.

18.5 TWO APPLICATIONS

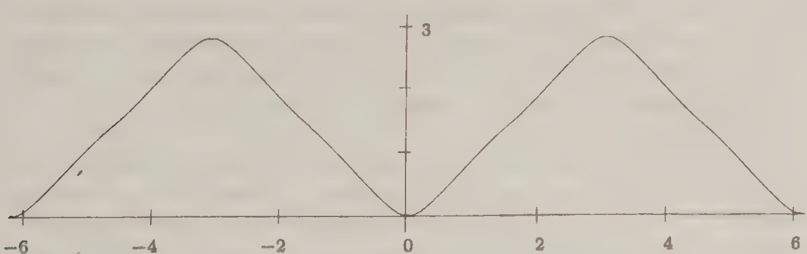
FOURIER SERIES: Consider the function f defined by

$$f(x) = \frac{\pi}{2} + \sum_{n=0}^{\infty} \frac{-4}{(2n+1)^2\pi} \cos((2n+1)x).$$

(The term $\pi/2$ is separated only because it doesn't fit the pattern of the others.) This series converges uniformly by the Weierstrass M-test, and the limit function is continuous by Theorem 15.4. What does the limit function look like? Here is part of the graph of the sum of the series up to $n = 3$:



This looks remarkably like the absolute value function! Of course, if we add up the periodic functions $\cos((2n+1)x)$, we expect the sum to be periodic. Here is the same partial sum over a larger domain:



So this is not exactly the absolute value function, but over an interval a bit larger than $[-3, 3]$ its resemblance to $|x|$ can't be denied. Is this any

more than an illusion? To evaluate this series for a particular value of x (other than $x = 0$) would be very difficult. We can, however, make some roundabout tests in which we compare $f(x)$ and $|x|$. Since the series defining f converges uniformly, Theorem 18.7 says we can integrate it term by term. Doing so, we find that

$$\int_{-\pi}^{\pi} f(x)\,dx = \pi^2 = \int_{-\pi}^{\pi} |x|\,dx$$

[since $\int_{-\pi}^{\pi} \cos(nx)\,dx = 0$ for $n = 1, 2, \ldots$]. Of course, the fact that the integrals of two functions are the same says precious little about the functions themselves. Now, if we multiply each term in the series for f by a bounded, continuous function, the series will still converge uniformly and we can still integrate the result term by term. For instance,

$$\int_{-\pi}^{\pi} f(x)\cos(x)\,dx = 2 = \int_{-\pi}^{\pi} |x|\cos(x)\,dx.$$

Pursuing this idea further, we find that

$$\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = 2 = \int_{-\pi}^{\pi} |x|\cos(nx)\,dx.$$

for $n = 1, 2, \ldots$. [To do this calculation, note that $\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = 0$ if $m \neq n$, while $\int_{-\pi}^{\pi} \cos^2(nx)\,dx = \pi$ for all n.]

We still can't say with certainty that $f(x) = |x|$ for $-\pi < x < \pi$, but we have "tested" both functions in infinitely many ways and gotten the same result every time. This seems to be very strong evidence (if you need more, show that

$$\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \int_{-\pi}^{\pi} |x|\sin(nx)\,dx = 0.$$

for $n = 1, 2, \ldots$). The series defining $f(x)$ is called the **Fourier series** of $|x|$. The question of the relationship between a function and its Fourier series is a study in itself and is not entirely settled.[2] You will see how to get from a function to its Fourier series in Exercise 18.5.1.

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[2] It is only a small exaggeration to say the the study of Fourier series gave rise to most of modern analysis. Cantor was led to his concept of cardinality in part by his study of a question about Fourier series, for instance. Fourier himself had the distinction (if it can be called that) of having been condemned to death by both sides in the French Revolution. Neither sentence seems to have been carried out.

THE ANTIDERIVATIVE OF A POWER SERIES: Combining the results of Theorems 15.15 and 18.7, we obtain:

COROLLARY 18.11: *If the series $\sum a_n(x-a)^n$ has radius of convergence R and $[b,c] \subseteq (a-R, a+R)$, then the integral of the series over this interval is the limit of the integrals of its partial sums.* ∎

EXAMPLES 18.5: 1. The function $f(x) = e^{-x^2}$ is of overwhelming importance in the study of statistics. A student of that subject is often called upon to discuss the relative sizes of integrals of f over various sets. We learn in calculus, though, that this function does not have an antiderivative that can be expressed easily. On the other hand, since f is continuous, the Second Fundamental theorem tells us that $F(x) = \int_0^x e^{-t^2}\,dt$ is an antiderivative of f. We may not be able to say much about $F(x)$, but we can write its Maclaurin series. Since $e^t = 1 + t + t^2/2 + t^3/6 + \ldots$, and this series has an infinite radius of convergence, we have

$$\begin{aligned} & e^{-t^2} \\ = \;& 1 + (-t^2) + \frac{(-t^2)^2}{2} + \frac{(-t^2)^3}{6} + \cdots \\ = \;& 1 - t^2 + \frac{t^4}{2} - \frac{t^6}{6} + \cdots \end{aligned}$$

for all t. Antidifferentiating this, we have $F(x) = x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots$.

EXERCISES 18.5

1. (a) Evaluate all the integrals mentioned in the section about Fourier series.

   (b) The Fourier series of a function $f(x)$ having period 2π is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

   where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$ (the π in the denominator is there for technical reasons). Find the Fourier series of the function given on $[-\pi, \pi)$ by

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 \le x < \pi \end{cases}$$

   and extended to the real line by saying $f(x + 2\pi) = f(x)$ for all x.

   (c) The series you found in (b) *can't* converge uniformly. Why?

2. Find the Fourier series for the functions $f(x) = x$ and $f(x) = x^2$.

3. If we let $f(x) = x$ for $0 \le x \le \pi$, then the absolute value function is the "even" extension of f to $[-\pi, \pi]$ and $y = x$ is the "odd" extension. We have found the Fourier series for these two functions, but they are quite different. Discuss this.

4. Find the Maclaurin series for $F(x) = \int_0^x \cos(t^2)\, dt$.

5. (a) Find the Maclaurin series for $f(x) = \frac{1}{1+x^2}$.

   (b) Find the Maclaurin series for $\arctan(x) = \int_0^x \frac{1}{1+t^2}\, dt$.

   (c) Describe the set over which the calculation in (b) is valid.

   (d) Recall that $\arctan(1) = \pi/4$. Find a series for π.

   (e) Note that 1 is an endpoint of the interval of convergence of the series in (b). Is the representation in (d) valid?

   (f) Show that the series in (d) converges conditionally. (This fact has given rise to a booming cottage industry –the finding of longer and longer decimal expansions of π.)

   (g) How many terms of the series in (d) would have to be added to produce an approximation for π accurate to an error of less than 10^{-100}?

   (h) What would be the practical difficulties involved in performing the calculation in (g) on a computer?

   (i) Why is there no cottage industry devoted to producing decimal expansions of e?

18.6 INTEGRALS WITH A PARAMETER

Our final examples involve functions defined by integrals. Suppose that f is defined on some rectangle $[a, b] \times [c, d]$ and consider $F(t) = \int_a^b f(x, t)\, dx$. In this context, the variable t is called a **parameter**. We will show that under certain conditions F is continuous, and then turn our attention to the (somewhat more interesting) question of the derivative of F.

THEOREM 18.12: *If f is continuous on $[a, b] \times [c, d]$ and F is defined as above, then F is uniformly continuous on $[c, d]$.*

PROOF: We will have to accept that some theorems we have proved for the real line also hold in the plane. The square $[a, b] \times [c, d]$ is a compact subset of $\mathbf{R}^2$, and so f is uniformly continuous. Thus, given $\varepsilon > 0$, there

is a $\delta > 0$ so that, for any $x \in [a, b]$, $|f(x, t_1) - f(x, t_2)| < \varepsilon$ whenever $|t_1 - t_2| < \delta$. Then

$$\begin{aligned} &|F(t_1) - F(t_2)| \\ &= \left| \int_a^b f(x, t_1) - f(x, t_2)\, dx \right| \\ &\leq \int_a^b |f(x, t_1) - f(x, t_2)|\, dx \\ &\leq \varepsilon(b - a), \end{aligned}$$

and consequently F is uniformly continuous (since this estimate does not depend on t_1 or t_2). ∎

THEOREM 18.13: *If f and $\frac{\partial f}{\partial t}$ are continuous on $[a, b] \times [c, d]$ and F is as above, then F is differentiable on (c, d) and $F'(t) = \int_a^b \frac{\partial}{\partial t} f(x, t)\, dx$.*

PROOF: Since we have a candidate for the derivative, we should examine the expression $f(u) - f(t) - \left[\int_a^b \frac{\partial}{\partial t} f(x, t)\, dx\right](u - t)$, which should be $o(u - t)$. Inserting the definition of F, this becomes:

$$\int_a^b f(x, u)\, dx - \int_a^b f(x, t)\, dx - \left[\int_a^b \frac{\partial}{\partial t} f(x, t)\, dx\right](u - t).$$

Observe that the expression $(u - t)$ may be moved inside the last integral since it doesn't depend on x. Having done this, we can combine the three integrals into one to obtain

$$\int_a^b \left[f(x, u) - f(x, t) - \frac{\partial}{\partial t} f(x, t)(u - t) \right] dx.$$

The integrand, as a function of u and t, is $o(u - t)$ (this is the definition of the partial derivative). The entire expression is continuous as a function of x, uniformly so in u and t. Thus it is integrable, and its integrals are uniformly bounded, say by M. It follows that the integral is not larger than $M[o(u - t)] = o(u - t)$, as desired. ∎

EXERCISES 18.6

1. (a) Verify that Theorem 18.13 holds if f is a polynomial of two variables.

   (b) Experiment with more complicated functions; trigonometric functions, exponentials, and so on.

2. Suppose $f : [a, b] \times [c, d] \to \mathbf{R}$ is continuous and is constant in its second variable for any value of the first variable [that is, $f(x, t_1) = f(x, t_2)$ for any x, t_1, and t_2].

   (a) Show that the function $F(t) = \int_a^b f(x, t)\, dx$ is constant.

   (b) Prove this using Theorem 18.13.

   (c) Prove this without referring to Theorem 18.13.

3. Suppose $f : [a, b] \times [c, d] \to \mathbf{R}$ and $f(x, t)$ satisfies the partial differential equation $Au_{tt} + Bu_t + Cu = D$ (where A, B, C, and D are constants).

   (a) Show that the function $F(t) = \int_a^b f(x, t)\, dx$ satisfies the ordinary differential equation $Ay'' + By' + Cy = D(b - a)$.

   (b) Consider this result without worrying about hypotheses on the function f.

   (c) Carefully describe the conditions f must satisfy to make this true.

18.7 "THE MOORE THEOREM"

There is a pattern in all this, having something to do with uniformity of one of the limit processes. Can this be generalized? We will state, but not prove, such a generalization. The proof may be found in *The Theory of Functions of Real Variables* by Lawrence M. Graves (published by McGraw-Hill Book Company in 1946).([3])

THEOREM 18.14: *Suppose the functions $f(x, y)$, $g(x)$, and $h(y)$ are all finite-valued and that $\lim_{x \to a} f(x, y) = h(y)$ on T and $\lim_{y \to b} f(x, y) = g(x)$ uniformly on S. Then the limits $\lim_{x,y \to a,b} f(x, y)$, $\lim_{x \to a} g(x)$, and $\lim_{y \to b} h(y)$ all exist and are equal and finite.* ∎

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[3] This text was once recommended to me with the statement, "If it's not in Graves, it's not true." "The Moore Theorem" is Theorem 2 in Chapter VII and is accompanied by my candidate for the most obscure footnote reference ever written:

See E. H. Moore, "Lectures on Advanced Integral Calculus" (unpublished), University of Chicago, Autumn Quarter, 1900. Manuscript in University of Chicago library, worked out by Oswald Veblen ....

As Casey Stengel used to say, you can look it up!

EXERCISES 18.7

1. In this exercise, we adapt the least squares method to the problem of derivatives. The **least-squares** (or **regression**) line for the finite set of points $\{(x_i, y_i) : i = 1, \dots, n\}$ is the line $y = mx + b$, where m and b are chosen to minimize the expression $\sum_{i=1}^{n}(y_i - (mx_i + b))^2$ (m and b are found by setting partial derivatives to 0). The regression line is supposed to "look like" the set of points in some way. Presently we will choose $m(h)$ and $b(h)$ so as to minimize

$$\int_x^{x+h} \left(f(t) - m(h)t - b(h)\right)^2 \, dt$$

and let $Lf(x) = \lim_{h\to 0} m(h)$ if this limit exists. This is certainly the slope of a line that looks like f in a certain way.

(a) Suppose f is continuous. Find formulas for $m(h)$ and $b(h)$. Discuss the interchange of integrals and partial derivatives in this calculation.

(b) If $f(x) = x^2$, show that $Lf(a) = 2a$ for all a.

(c) Show that if f is differentiable (in the ordinary sense), then f has a derivative in this sense and $f'(x) = Lf(x)$.

(d) Find a function that is not differentiable (in the ordinary sense) but is differentiable in this sense.

(e) Show that if f is differentiable in this sense and $c \in \mathbf{R}$, then cf is differentiable in this sense, and $L(cf) = cL(f)$.

(f) Show that if f and g are differentiable in this sense, then $f + g$ is differentiable in this sense, and $L(f + g) = L(f) + L(g)$.

(g) Does the Product Rule work for this derivative? Is it necessarily that case that $L(fg)(a) = f(a)L(g)(a) + g(a)L(f)(a)$?([4])

2. Reconsider Exercise 15.5.5 in light of the results of this chapter.

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[4] This idea is examined in "A new extension of the derivative" *American Mathematical Monthly* 97 (1990), 230–233, by Daniel B. Kopel and the author. It was first developed as Mr. Kopel's undergraduate thesis.

Part Four

Selected Shorts

In the final part of the book we will look at three short subjects and then plug a lingering gap in our knowledge. In the next three chapters, we consider questions that are very different from those to which we have become accustomed. Most of our time in mathematics classes is spent proving statements of the form "If X does Y, then X must also do Z." In this part of the book, we examine questions "If X does Y, how badly can X *fail* to do Z?"

In Chapter 20 we ask "If a function is increasing, at how many points can it fail to be continuous?" This may be the least natural question of the three, and the fact that we can give a complete answer may come as a surprise. Next we ask "If a function is continuous, at how many points can it fail to be differentiable?" We will find that we spent all our time in calculus talking about functions that, in a very precise sense, are hardly there at all! Finally we consider the question "If a function is *integrable*, at how many points can it fail to be continuous?" This is the deepest question of the three, and we will fall just short of finding a complete answer to it.

These questions help us see how the properties described in the Big Theorem affect what we can and can't do with calculus and show us how delicately the whole subject is strung together. There is a subtle common theme to these questions that gives us some hints about directions we might go in the subject.

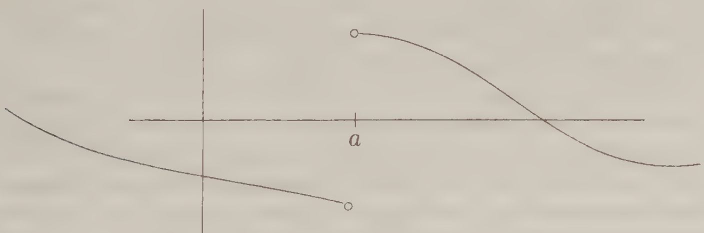
Finally, we dispose of a problem that has been begging our attention from the start. We now know a lot about how the real numbers work, but *we still don't know what they are*! In Chapter 22 we will, at long last, build the real number system. Much of this book has been designed to shake us in our certainty that we know at least what the real numbers are. In Chapter 22 our understanding will be brought full circle. What better way to finish?

Chapter 19

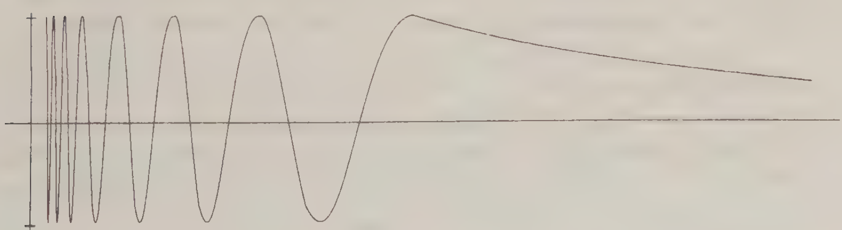
Increasing Functions

19.1 DISCONTINUITIES

We spend so much time and effort studying continuous functions and their properties that it might never occur to us that discontinuities could have something interesting to teach us, too. In calculus, we saw two basic types of discontinuities. First, we encountered the type we will now call a **jump**, which might look like this:



Later we encounter the more dramatic behavior of a function like $f(x) = \sin(1/x)$, which is discontinuous at $x = 0$ despite the fact that there is no obvious break in the graph. The graph looks something like this for $0 < x < 2$, but it oscillates so quickly as x gets near 0 that it is very difficult to depict accurately:



An even more extreme example of this "up-and-down" behavior is found

in the Dirichlet function (which is *impossible* to graph):

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

After we describe them precisely, we will find that these two cases (the jump and the bouncing up and down) are, in a precise sense, the only possibilities!

DEFINITION 19.1: (a) Suppose $f : A \to \mathbf{R}$ and that A is a neighborhood or deleted neighborhood of a.([1]) Then f has a **jump discontinuity** at a if the one-sided limits $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ both exist but are different. This is also called a **simple discontinuity** or (in contrast to the next definition) a **discontinuity of the first type**.
(b) Suppose $f : A \to \mathbf{R}$ and that there is a $\delta > 0$ so that either $(a, a+\delta)$ or $(a-\delta, a)$ is contained in A.([1]) Then f has a **discontinuity of the second type** at the point a if either $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x)$ fails to exist.

This terminology is quite bland, but we're stuck with it. The observant reader has noticed that there is a third possibility not accounted for in these definitions. It might happen that $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$ [that is, $\lim_{x\to a} f(x)$ exists], but f is discontinuous. Such a discontinuity can be "removed" by redefining the function, setting $f(a) = \lim_{x\to a} f(x)$. *We will assume that all such* **removable discontinuities** *have been dealt with in this way* and will not concern ourselves with them further. We will also concentrate for now only on *bounded* functions since unbounded functions present different (and less interesting) problems.

The Dirichlet function and the function $f(x) = \sin(1/x)$ both display discontinuities of the second type (the former at every real number, the latter only at $x = 0$). These functions are very different, which might lead us to think that such discontinuities are intractable. Fortunately, they are more manageable than we might fear. The following lemma says that a function with a discontinuity of the second type *must* display the up-and-down behavior apparent in $\sin(1/x)$ and the Dirichlet function.

LEMMA 19.2: *Suppose $f : A \to \mathbf{R}$ is bounded and there is a $\delta > 0$ such that $(a, a+\delta) \subseteq A$. Then $\lim_{x\to a^+} f(x)$ fails to exist if and only if there are numbers $c < d$ so that for every $0 < \varepsilon < \delta$ there are numbers x and y in $(a, a+\varepsilon)$ with $f(x) < c$ and $f(y) > d$. A similar statement holds if $\lim_{x\to a^-} f(x)$ fails to exist.*

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[1] This condition only serves to ensure that the limits in the definition make sense.

PROOF: Since $\lim_{x\to a^+} f(x)$ fails to exist, there is a decreasing sequence (x_n) with $\lim x_n = a$ but such that $\lim f(x_n)$ does not exist. Since f is bounded, the sequence $(f(x_n))$ is also bounded. Then $(f(x_n))$ is a bounded, divergent sequence, and so it has subsequences that converge to two different limits, say γ and δ. We may assume that $\gamma < \delta$. Any numbers c and d with $\gamma < c < d < \delta$ will satisfy the theorem (be sure you see why this is so). ∎

EXAMPLES 19.1: 1. Let $D(x)$ be the Dirichlet function, $c = 1/4$, and $d = 3/4$. For any $a \in \mathbf{R}$ and any $\varepsilon > 0$, if $x \in (a, a+\varepsilon)$ is rational and $y \in (a, a+\varepsilon)$ is irrational, then $D(x) > d$ and $D(y) < c$.

2. For the function $f(x) = \sin(1/x)$, we can take $c = -1/2$ and $d = 1/2$. Then x can be any number of the form $1/(2n+3/2)\pi$ [at each of which $\sin(1/x) = -1$], and y can be any number of the form $1/(2n+1/2)\pi$ [where $\sin(1/x) = 1$]. There is one of each of these points in any interval $(0, \varepsilon)$.

3. Lemma 19.2 does not hold if the function is not bounded. Consider $f(x) = 1/x$. Though $\lim_{x\to 0^+} f(x)$ fails to exist, the conclusion of the theorem does not hold.

EXERCISES 19.1

1. Complete the proof of Lemma 19.2.

2. Show that a function having a discontinuity of the second type can't be of bounded variation (see Exercise 17.7.6).

3. Suppose that f has a discontinuity of the second type as $x \to a$ from the right, but is continuous otherwise. Show that the arc length of the graph of f is infinite in any interval $(a, a+\varepsilon)$. (Don't worry about the technicalities of the definition of arc length now.)

19.2 DISCONTINUITIES OF MONOTONE FUNCTIONS

The next two results show that the discontinuities of monotone functions and the behavior of such functions near a discontinuity are very predictable.

THEOREM 19.3: *A monotone function can have only jump discontinuities.*

PROOF: We need only observe that the condition of Lemma 19.2 can't occur for a monotone function. Suppose f is increasing, $x > a$, and c and d are such that $f(x) < c < d$; then $f(y) \le f(x) < c$ for all $y \in (a, x)$, and it cannot be the case that $f(y) > d$ for any such y. ∎

LEMMA 19.4: *If f is an increasing function having a discontinuity at a, then $\lim_{x\to a^+} f(x) > \lim_{x\to a^-} f(x)$. The inequality is reversed for a decreasing function.*

PROOF: Since the discontinuity must be a jump, both one-sided limits exist. We will show first that $\lim_{x\to a^+} f(x) \ge \lim_{x\to a^-} f(x)$. Let $\varepsilon > 0$ be given. Since both one-sided limits exist, we may choose $x > a$ so that $f(x) - [\lim_{x\to a^+} f(x)] < \varepsilon/2$ and $y < a$ so that $[\lim_{x\to a^-} f(x)] - f(y) < \varepsilon/2$. Then

$$\begin{aligned} & \lim_{x\to a^+} f(x) - \lim_{x\to a^-} f(x) \\ > \; & \left(f(x) - \frac{\varepsilon}{2}\right) - \left(f(y) + \frac{\varepsilon}{2}\right) \\ = \; & f(x) - f(y) - \varepsilon \\ \ge \; & -\varepsilon. \end{aligned}$$

The last inequality holds because f is increasing and $y < x$. This is true for any $\varepsilon > 0$. By Exercise 4.6.13, $\lim_{x\to a^+} f(x) - \lim_{x\to a^-} f(x) \ge 0$. Since we are assuming f doesn't have a removable discontinuity at a, it can't be the case that $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$, and strict inequality holds. ∎

THEOREM 19.5: *The set of points at which a monotone function is discontinuous is countable.*

PROOF: Suppose f is increasing. If f has a discontinuity at the point a, let $I_a = (\lim_{x\to a^-} f(x), \lim_{x\to a^+} f(x))$. By Lemma 19.4, $I_a \ne \emptyset$. We will show that if f has discontinuities at a and b, then $I_a \cap I_b = \emptyset$. Suppose $a < c < b$. Since f is increasing,

$$\lim_{x\to a^+} f(x) \le f(c) \le \lim_{x\to b^-} f(x).$$

Thus $y < f(c)$ for $y \in I_a$ and $y > f(c)$ for $y \in I_b$. It follows that $I_a \cap I_b = \emptyset$. This means that the association $a \to I_a$ is one-to-one. In the proof of Theorem 8.11, we showed that any collection of mutually disjoint nonempty open intervals is countable, so $\{a : f \text{ has a discontinuity at } a\}$ is countable. ∎

EXERCISES 19.2

1. Show that a composition of increasing functions is increasing.

2. Draw a sketch to illustrate the proof of Theorem 19.3.

3. Complete the proof of Lemma 19.4.

4. (a) Show that the association $a \to I_a$ in the proof of Theorem 19.5 is one-to-one.

   (b) We haven't checked that the the association $a \to I_a$ is onto. How can we draw the conclusion at the end of the proof?

5. Show that the discontinuities of a *decreasing* function are all jumps and that there can be at most countably many of them.

19.3 MORE ON JUMPS

The proof of Theorem 19.5 rests heavily on the fact that the function involved is increasing. But it is not so much the nature of increasing functions as it is the nature of jumps that makes the result true. In contrast to the following theorem, recall that the Dirichlet function has a discontinuity of the second type at *every* point of the real line.

THEOREM 19.6: *Any function can have only countably many jump discontinuities.*

PROOF: We show first that if f has a jump discontinuity at a, it can't have another one as big nearby. Suppose f has a jump discontinuity at a and let $|\lim_{x\to a^+} f(x) - \lim_{x\to a^-} f(x)| = \varepsilon$ (we will call this the **jump** of f at a). There is a $\delta > 0$ so that $|f(x) - \lim_{x\to a^+} f(x)| < \varepsilon/2$ when $x \in (a, a+\delta)$. Then for $x, y \in (a, a+\delta)$, we have $|f(x) - f(y)| < \varepsilon$, and there can be no jump as big as ε in the interval $(a, a+\delta)$. A similar argument holds in an interval $(a-\delta, a)$ (though δ may be different). Thus if f has a jump of ε at a, there is a $\delta > 0$ so that f has no jump of more than ε in the interval $(a-\delta, a+\delta)$. Now suppose a_1 and a_2 are points where f has a jump of at least ε (with corresponding δ_1 and δ_2). Note that $(a_1 - \delta_1/2, a_1 + \delta_1/2)$ and $(a_2 - \delta_2/2, a_2 + \delta_2/2)$ are disjoint. The points where f has a jump of ε or more can each be enclosed in an open interval, and these intervals can be chosen to be mutually disjoint. Thus there are only countably many such points. Thus for each natural number n, there are countably many points where f has a jump discontinuity with jump more than $1/n$. But *any* jump is larger than $1/n$ for some n (why?), so

$$\{a : f \text{ has a jump discontinuity at } a\}$$
$$= \bigcup_{n=1}^{\infty}\{a : f \text{ has a jump of more than } 1/n \text{ at } a\}.$$

We have written the set of points where f has a jump discontinuity as a countable union of countable sets, and so this set is countable. ∎

Notice that Theorem 19.5 follows directly from Theorems 19.3 and 19.6. It is sometimes easier to show that the discontinuities of a function must be jumps than to count them directly (see Exercise 19.1.2).

EXERCISES 19.3

1. If f has the property that $|f(x) - f(y)| < \varepsilon$ for all x and y, show that f can't have a jump discontinuity with a jump of more than ε.

2. Suppose that f has only jump discontinuities. If f has a jump at a, write $J_a = \lim_{x\to a^+} f(x) - \lim_{x\to a^-} f(x)$ and let $J(x) = \sum_{a\le x} J_a$.

   (a) Show that J is continuous at any point where f is continuous.

   (b) Show that J is constant on any interval on which f is continuous and has a jump discontinuity at each point where f has a jump discontinuity. A function whose graph consists of intervals of constancy separated by jump discontinuities is called a **jump function**.

   (c) Show that the function $f(x) - J(x)$ is continuous. Thus every function having only jump discontinuities can be written as the sum of a continuous function and a jump function.

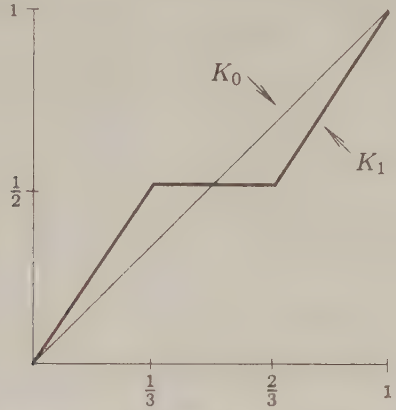
19.4 THE CANTOR FUNCTION

We finish this chapter by looking at another example of curious behavior in an increasing function. Here we will review the construction of the Cantor set (Exercise 8.4.7), while at the same time building a very strange function. We begin with a lemma, whose proof is left as Exercise 19.4.1.

LEMMA 19.7: *If (f_n) is a sequence of functions converging to f, and if f_n is increasing for each n, then f is increasing.* ∎

We now begin our construction. The construction of the Cantor set begins with the interval $C_0 = [0, 1]$. At each step in the process, we find ourselves with a collection of closed intervals. To get to the next stage, we remove the open middle third of each interval. Now let $K_0(x) = x$ for $0 \le x \le 1$. Removing the middle third of C_0, we obtain $C_1 = [0, 1/3] \cup [2/3, 1]$. It is

easiest to define K_1 by showing its graph:



(It's not difficult to produce formulas for K_1, but the effort would only get in the way at this point.) Note that K_1 is constant on the interval removed from C_0 to make C_1 and that the maximum difference between K_0 and K_1 is 1/6, occurring at 1/3 and 2/3. To construct $K_2, K_3, \ldots$, we replace each nonhorizontal portion of the previous one with a piece resembling K_1. For example, here is the graph of K_2:

Observe that K_2 is constant on the interval on which K_1 is constant and on the intervals removed from C_1 to make C_2, and that the maximum difference between K_2 and K_1 is $1/12$. We continue in this way with each function K_n having the following properties: K_n is continuous and increasing; K_n is constant on the intervals on which K_{n-1} is constant and on all the intervals removed from C_{n-1} to make C_n; the maximum

difference between K_n and K_{n-1} is $1/(6 \times 2^{n-1})$. From the last of these properties, it follows that for $m > n$,

$$\begin{aligned} & \max |K_n(x) - K_m(x)| \\ \leq \quad & \frac{1}{6}\left(\frac{1}{2^n} + \cdots + \frac{1}{2^{m-1}}\right) \\ < \quad & \frac{1}{6 \times 2^{n-1}} \end{aligned}$$

By Theorem 15.6, (K_n) converges uniformly. Call its limit K. Since K_n is continuous for all n, K is continuous. By Lemma 19.7, K is increasing. Furthermore, K is constant on every open interval removed from C_0 to make the Cantor set; that is, $K'(x) = 0$ for x in any of these intervals.

We've constructed a continuous function whose derivative is 0 except on a set whose length is 0 (a precise definition of "length 0" is given in Chapter 21). Yet the value of K manages somehow to change from 0 to 1. It seems that we have gotten from 0 to 1 without ever moving! A function like K is called, appropriately enough, **singular**. Singular functions have properties even more peculiar than the one we've seen. The function $K'(x)$ is continuous (and equal to 0) except for a set of length 0 (the Cantor set). As we shall see later, this means that the *integral* of $K'(x)$ is zero over any interval. But K is not identically 0, and so K is not the integral of its derivative. What is the derivative of K? Because of the Mean Value theorem, K can't be differentiable in the usual sense at the points of the Cantor set. The slanted sections of K_n have slope $(3/2)^n$, and it would make sense to say that the derivative of K is $+\infty$ at all points of the Cantor set (we would have to adjust our theory of differentiation to allow infinite derivatives). It can be shown that the derivative of any continuous, singular function must be $\pm\infty$ at any point where it isn't 0.

EXERCISES 19.4

1. Prove Lemma 19.7.

2. Give a precise definition of the function K_1.

3. Verify the claims made about the functions K_n.

4. Let $f : [a, b] \to \mathbf{R}$ be such that f' is integrable.[2] Recall that the function given by $F(x) = \int_a^x f'(t)\,dt$ is absolutely continuous (see

[2] Remember that f' need not be defined everywhere in order for it to be integrable.

Exercise 17.7.7). Show that the function $f(x) - F(x)$ is singular. Every function whose derivative is integrable can be written as the sum of a singular function and an absolutely continuous one.

5. Discuss the results of constructing a "Cantor-like" function based on the fat Cantor set of Exercise 8.4.7.i.

6. (a) Construct an increasing singular function (similar to the Cantor function) that increases from 0 to 1 on the interval $[a, b]$. Call this function $f_{[a,b]}$. ($f_{[0,1]}$ should be the Cantor function.)

   (b) Show that $f_{[0,1]} + f_{[\frac{1}{3},\frac{2}{3}]}(x)$ is singular. What are its intervals of constancy?

   (c) Construct a *strictly* increasing singular function.

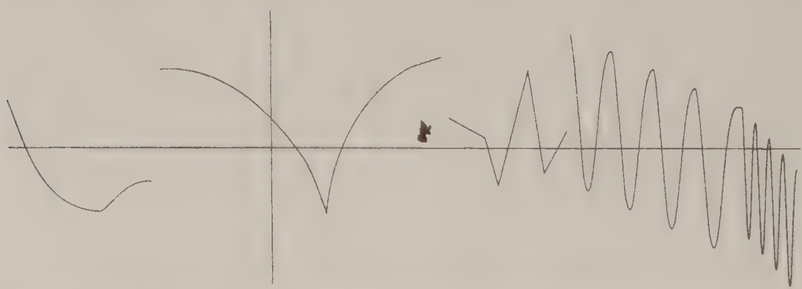
7. How does the Mean Value theorem justify the statement in the last paragraph of the section?

Chapter 20

Continuous Functions and Differentiability

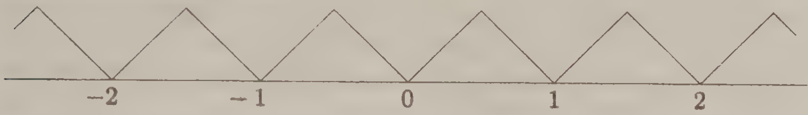
20.1 SEPARATING THE GOOD FROM THE BAD

As we learn about functions, the image evoked by the word changes. When we are young, a "function" is, we think, a straight line. Later we learn about curves, and then discontinuities. Finally we learn about differentiability, and our response to the request "draw a graph" might look something like this:



This graph has points of discontinuity or nondifferentiability separated by intervals in which the function is differentiable. Is this as complicated as a function can get? We know, of course, that a function can have many more discontinuities than this. The Dirichlet function is discontinuous *everywhere*. But what about the points of nondifferentiability? At how many points can a continuous function fail to be differentiable? We must be a little careful about how we phrase this question and what sort of answer we expect. We might, for example, specify a set and ask whether there is a function that is differentiable (or not) precisely on that set. This is a difficult task, and we will not begin to attack it here. But we can easily find a function that fails to be differentiable on an infinite set. For instance, the function whose graph is below fails to be differentiable at $n/2$ for each integer n. Though the set of points of nondifferentiability

of this function is infinite, its graph still has "bad" points separated by intervals of "good" ones.



Can a continuous function be any worse-behaved than this? We will find an answer so dramatic that we can stop worrying about exactly what the question was! We will construct a function that is continuous *everywhere* but differentiable *nowhere*. An example of a function like this (which you will examine in Exercise 20.3.2) was found by Weierstrass in the late nineteenth century. It caused quite a stir in the mathematical world, forcing as it does a thorough reconsideration of most of the ideas involved. The function we will construct was described by van der Waerden in 1930. It has a pictorial appeal that we will find to be, unfortunately, largely misleading. Still, being led down a garden path can sometimes be educational.

20.2 THE NATURE OF CORNERS

We often say that a differentiable function looks like a straight line as we look at it more and more closely. The absolute value function—the first example we see of a nondifferentiable function—has a "corner" at $x = 0$. But how do we identify a corner? A graph is a straight line if the quotient $(b-d)/(a-c)$ is the same for any two points on the graph (a, b) and (c, d). Consider the absolute value function. In any neighborhood of 0 we can select two points with positive first coordinates. The difference quotient formed using them will be 1. Any difference quotient formed using two points with negative first coordinates is -1. Within any neighborhood of 0, we can form difference quotients that are far apart.

EXAMPLES 20.2: The description of how to find a corner in the previous paragraph seems precise enough. It is, however, fallacious. The function given by

$$f(x) = \begin{cases} x^2 & x \in \mathbf{Q} \\ 0 & x \notin \mathbf{Q} \end{cases}$$

is differentiable at 0 and $f'(0) = 0$. But we can chose points near and to the right of 0 to produce a quotient that is just about anything we like! The same can be done to the left of 0. The picture below indicates why this is so. (Though it is impossible to draw this graph well, we can get the idea of the problem. You will examine this claim about secant lines

in Exercise 20.2.1.)

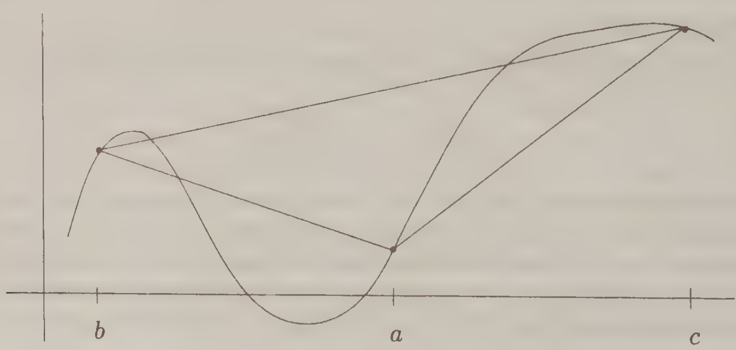
It seems, though, that the behavior of secant lines near a point where a function is differentiable *must* be predictable. In the this example we have been a little too lax about how the secant lines were chosen. The following lemma, a sort of Cauchy criterion for derivatives, clears this up.

LEMMA 20.1: *The function f is differentiable at the point a if and only if the following condition holds: For each $\varepsilon > 0$, there is a $\delta > 0$ so that if x_1, x_2, x_3, and x_4 are elements of $(a-\delta, a+\delta)$ with x_1 and x_2 on opposite sides of a and x_3 and x_4 on opposite sides of a, then*

$$\left|\frac{f(x_1)-f(x_2)}{x_1-x_2}-\frac{f(x_3)-f(x_4)}{x_3-x_4}\right|<\varepsilon.$$

The phrase "on opposite sides of a" is to be interpreted in the following sense: If $x_1 < a$, we must have $x_2 \geq a$, and if $x_1 > a$, we must have $x_2 \leq a$. If $x_1 = a$ or $x_2 = a$, then x_1 and x_2 are on opposite sides of a regardless of where the other is. Of course, we can always assume that $x_1 < x_2$ if it is convenient to do so.

PROOF: We begin with a question about secant lines. If $b < a < c$, can the slope of the secant line between $(b, f(b))$ and $(c, f(c))$ be computed in some easy way from the slopes of the two secant lines passing through $(a, f(a))$?



In this picture, the slope of the secant line over $[b, a]$ is less than the slope of the secant line over $[b, c]$, and the slope of the secant line over $[a, c]$ is greater than the slope of the secant line over $[b, c]$. Perhaps the slope of the longer secant line is the average of the other two? Almost any example will show that this is not the case. However, the slope of the longer line is a *weighted* average of the other two, and with a little algebra, we can find the appropriate weights:

$$\frac{f(c)-f(b)}{c-b} = \frac{f(a)-f(b)}{a-b}\frac{a-b}{c-b} + \frac{f(c)-f(a)}{c-a}\frac{c-a}{c-b}.$$

If f is differentiable, the large fractions on the right can be made close to $f'(a)$ by making b and c sufficiently close to a. More precisely:

$$\begin{aligned} & \frac{f(c)-f(b)}{c-b} \\ = {} & [f'(a)+\nu]\frac{a-b}{c-b} + [f'(a)+\mu]\frac{c-a}{c-b} \\ = {} & f'(a) + \nu\frac{a-b}{c-b} + \mu\frac{c-a}{c-b}. \end{aligned}$$

Where ν and μ approach 0 as b and c approach a. The "only if" part of the theorem follows from this. The other part of the proof is similar to Exercise 8.6.5, and is left as Exercise 20.2.2. ■

EXERCISES 20.2

1. (a) If $f(x) = x^2$ for $x \in \mathbf{Q}$ and $f(x) = 0$ for $x \notin \mathbf{Q}$, show that for any $\varepsilon > 0$, $\delta > 0$, and $s \in \mathbf{R}$, there are numbers $a, b \in (0, \varepsilon)$ with

$$\left|\frac{f(b)-f(a)}{b-a} - s\right| < \delta.$$

   (b) Explain why this means that we can find secant lines whose slopes are "just about anything we like."

   (c) Show that f is differentiable at 0 and that $f'(0) = 0$.

   (d) Show directly that f satisfies the conditions of Lemma 20.1.

   (e) Show that the function $g(x) = \sin x$ is differentiable at 0 by showing that it satisfies the conditions of Lemma 20.1.

2. (a) Complete the proof of Lemma 20.1.

   (b) How is the stipulation that x_1 and x_2 are on opposite sides of a used in the proof of this lemma?

3. (a) We have noted that the absolute value function is not differentiable

at 0, roughly because the slopes of its tangent lines make an abrupt change there. Explain this precisely by considering Exercise 12.6.2.

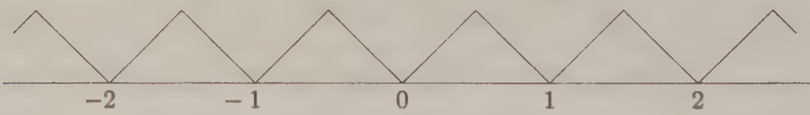
(b) But the function

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

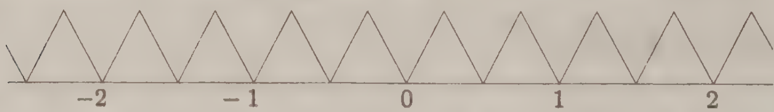
has a jump discontinuity, and it's the derivative of $|x|$, isn't it?

20.3 VAN DER WAERDEN'S FUNCTION

We now take another look at the function seen at the beginning of the chapter, which may be defined as $W(x) = \min\{|x - n| : n \in \mathbf{Z}\}$ [that is, $W(x)$ is the distance from x to the nearest integer]. Here again is the graph of W:



If we look at this graph near any point that is "half an integer," we can find secant lines with slopes of 1 and -1. In the graph of $W(2x)$, there are more such changes of slope and they are closer together:



We will want to keep the slopes of these segments ± 1. We can do this with a change of vertical scale. Here is the graph of $(1/2)W(2x)$:

Each function $(1/4)W(4x)$, $(1/8)W(8x)$, $\ldots$, fails to be differentiable on a set that is "twice as big" as the one before. (In the sense of cardinality, of course, these sets are all the same size.) None of these functions is *nowhere* differentiable, but they have more and more points of nondifferentiability and seem to get closer to what we want. We might guess that $\lim_{n\to\infty}(1/2^n)W(2^n x)$ would fill the bill. But this limit is 0 for all x (be sure you why), and the function $f(x) = 0$ is quite differentiable indeed. However, the functions $(1/2^n)W(2^n x)$ do provide us with secant lines of slope 1 and slope -1 very close together. We have only to assemble them in the right way.

THEOREM 20.2: *The function $\sum_{n=0}^{\infty} \frac{1}{2^n} W(2^n x)$ is continuous everywhere yet differentiable nowhere.*

PROOF: This function is continuous by the Weierstrass M-test. We will use Lemma 20.1 to show that it is nowhere differentiable. Consider the numbers $m/2^n$ for integer values of m and positive integer values of n. These are the dyadic rationals (see Exercise 6.1.13), which are a dense subset of the real line (so we can find them close to and on either side of any real number). Note that $W(2^n x) = 0$ for all x of the form $m/2^k$, where $n \geq k$. Also, if $k < n$, the slope of any secant line to $W(2^n x)$ based on adjacent dyadic rationals having the same denominator is 0, 1, or -1 (this is clear from the pictures). Thus the slope of any such secant line to $\sum_{n=0}^{\infty} \frac{1}{2^n} W(2^n x)$ is an integer. Pairs of adjacent dyadic rationals on opposite sides of any point can be chosen so that the difference between the slopes of the associated secant lines is not 0, hence this difference must be at least 1. By Lemma 20.1, this function is not differentiable at any point. ∎

EXERCISES 20.3

1. (a) With $W(x)$ as above, show that $\lim_{n\to\infty}(1/2^n)W(2^n x) = 0$.

   (b) Discuss the behavior of $\lim_{n\to\infty} W(2^n x)$.

2. (a) Show that the function $f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \cos(13^n \pi x)$ is continuous and nowhere differentiable. This is a version of the function given by Weierstrass. Notice that the functions that make up this series have no "corners" at all!

   (b) Suppose a is an odd natural number, $0 < b < 1$, and $ab > 1 + \frac{3}{2}\pi$. Show that the function $g(x) = \sum_{n=0}^{\infty} b^n \cos(a^n \pi x)$ is continuous and nowhere differentiable. (This is an involved proof. Here are some hints: (i) All the conditions on a and b will come into play *eventually*. (ii) Look at the standard difference quotient. (ii) Look first at a "finite" part of the series, and then at the corresponding "tail." (iv) If the finite part of the series goes to infinity, the only way the function can have a derivative is if the tail goes to $-\infty$; make sure the tail is bounded below. (v) When examining the tail of the series, let $a^k x = \alpha_k + \beta_k$, where α_k is an integer and $-1/2 \leq \beta_k < 1/2$, and look at $h = \frac{1-\beta_k}{a^k}$.)

   (c) (Research problem!) Suppose that f is a continuous periodic function having an interval on which it is strictly increasing and an interval on which it is strictly decreasing. Are there necessarily positive numbers a and b so that $\sum_{n=0}^{\infty} b^n f(a^n x)$ is nowhere differentiable?

20.4 SETS OF FIRST AND SECOND CATEGORY

We had to work pretty hard to get the van der Waerden function. This effort might lead us to believe that continuous, nowhere differentiable functions are very rare. This is not the case at all, and finding out why will lead us into some new territory.

There are many ways a set can be said to be small. A set with three elements is smaller than a set with four, a finite set is smaller than an infinite set, and a countable set is smaller than an uncountable one. Topologically, we might consider a set to be large if it is dense in the real numbers. In this sense, the rationals and irrationals are both large, and an idea of size that doesn't distinguish these two sets might not be useful. But we are on the right track.

DEFINITION 20.3: (a) A set D is **dense in the set S** if $S \subseteq D^-$ (that is, S is contained in the closure of D).

(b) A subset of the real line is said to be **nowhere dense** if it is not dense in any nonempty open interval.

EXAMPLES 20.4: 1. You will show in Exercise 20.4.3 that a union of finitely many nowhere dense sets is nowhere dense. It is easy to see that a set with one element is nowhere dense (it is its own closure and clearly contains no open interval). Thus any finite set is nowhere dense.

2. The set of natural numbers is also closed and contains no open interval, so $\mathbf{N}$ is nowhere dense, even though it is infinite.

3. The rational numbers and the irrational numbers are dense in *any* set. The rationals are small in the sense of cardinality yet big in this sense. Further, we see that an infinite union of nowhere dense sets can be dense. The set of rational numbers is a countable union of sets with one element.

4. The Cantor set (Exercise 8.4.7) is closed (and hence equal to its closure) yet contains no open interval. Though it is uncountable, the Cantor set is nowhere dense. It is big in the sense of cardinality but small in this sense. This is one reason the Cantor set is so interesting.

The third example above tells us where we should look next. The rational numbers and the irrational numbers are both dense in the whole real line, but they are different in that the set of rational numbers can be built from a countable union of nowhere dense sets. The set of irrational numbers, we will find, can't.

DEFINITION 20.4: A set is **of first category** if it can be written as a countable union of nowhere dense sets; otherwise, it is **of second category**. (These bland names are sometimes replaced by the more descriptive *meager* and *nonmeager*, respectively.)

Note that a set that is a countable union of first category sets is also of first category. The following theorem strengthens this observation.

THEOREM 20.5: (The Baire Category Theorem) *If $\{S_n : n = 1, 2, \ldots\}$ is a countable collection of closed sets whose union contains a nonempty open interval, at least one of the sets S_n must contain a nonempty open interval.*

Observe that the negation of the conclusion—*None of the sets S_n contains a nonempty open interval*—seems to give us more information than the conclusion itself. This means a proof by contradiction is appropriate. Beyond this beginning, we should notice that the proof merely takes advantage of the facts available. The appearance of smaller and smaller intervals suggests the Nested Intervals property (though some care is necessary to ensure that a nest of *closed* intervals can be produced).

PROOF: Suppose that none of the sets S_n contains a nonempty open interval but that $(a_0, b_0) \subseteq \cup_n S_n$. Since (a_0, b_0) is not contained in S_1 (this is the assumption), there exists $x_1 \in (a_0, b_0) \backslash S_1$. Since S_1 is closed and $x_1 \notin S_1$, x_1 is not a cluster point of S_1, and so we may find an open interval (a_1, b_1) containing x_1, disjoint from S_1, and such that the interval $[a_1, b_1]$ is contained in (a_0, b_0) [be sure you see why this last condition can be met]. Now (a_1, b_1) is not contained in S_2. We may repeat this argument to find $x_2 \in (a_1, b_1) \backslash S_2$ and (a_2, b_2) that contains x_2, is disjoint from S_2, and is such that $[a_2, b_2]$ is contained in (a_1, b_1).

Now $[a_1, b_1]$, $[a_2, b_2]$, $\ldots$, is a nest of closed bounded intervals, hence has a nonempty intersection by the Nested Intervals property. Now let $x \in \bigcap_n [a_n, b_n]$. Since $[a_n, b_n] \cap S_n = \emptyset$ and $x \in [a_n, b_n]$ for all n, we have $x \notin S_n$ for all n. But $x \in [a_1, b_1] \subseteq (a_0, b_0) \subseteq \bigcup_n S_n$, and this is a contradiction. ∎

COROLLARY 20.6: *The set of real numbers is of second category.*

PROOF: Suppose that the real numbers are written $\mathbf{R} = \bigcup_n S_n$. Then it is also the case that $\mathbf{R} = \bigcup_n S_n^-$, and so $\bigcup_n S_n^-$ is a countable union of closed sets that contains a nonempty open interval. By the Baire Category theorem, one of the sets S_n^- must contain a nonempty open interval. That is, at least one of the sets S_n cannot be nowhere dense. Thus $\mathbf{R}$ is not of first category. ∎

COROLLARY 20.7: *The set of irrational numbers is a second category subset of the real numbers.*

PROOF: Left as Exercise 20.4.5. ∎

The proof of Theorem 20.5 rests on the completeness of the real numbers (in the guise of the Nested Intervals property) yet draws from it a purely topological result. This is a good indication that something important is at stake here. The question of whether a topological space is of second category is of great interest in advanced applications, and spaces that are not of second category are very different from those that are. This is a major concern in the subject called functional analysis. One must avoid the impression that "the biggest set around" is always of second category. This is why Corollary 20.6 is important. We could, for instance, build a theory of calculus around the rational numbers without ever mentioning any bigger sets, but that wouldn't make $\mathbf{Q}$ a set of second category, and the "calculus" we would obtain would be very different from the one with which we are familiar.

EXERCISES 20.4

1. Show that a set that is dense in the real line is also dense in any subset of the real line.

2. Show that a set is nowhere dense if the interior of its closure is empty.

3. Show that a union of finitely many nowhere dense sets is nowhere dense.

4. (a) Show that a countable union of sets of first category is of first category.

   (b) Show that a subset of a set of first category is of first category.

5. Prove Corollary 20.7.

6. (a) If A is a first-category subset of B and B is of second category, show that $A \neq B$.

   (b) Show that irrational numbers exist.

7. Is the Baire Category theorem *equivalent* to the completeness of $\mathbf{R}$? (That is, does it imply a part of the Big Theorem?)

8. (a) In Exercise 8.4.7.i, you constructed a fat Cantor set (one whose length is not 0). Show that this set is nowhere dense.

(b) If $f(x) = px + q$, the set $f(S)$ is called a **linear translation** of S. Show that any linear translation of a nowhere dense set is nowhere dense.

(c) Given $a < b$, show that $[a, b]$ is a linear translation of $[0, 1]$.

(d) Construct (from the fat Cantor set) a subset of $[0, 1]$ that is of first category but whose length is 1 (we may assume that the length of a union of countably many disjoint sets is the sum of the lengths of the individual sets).

20.5 THE SET OF DIFFERENTIABLE FUNCTIONS

The proof of Lemma 20.8 is left as Exercise 20.5.1.

LEMMA 20.8: *If S is a closed set whose complement is everywhere dense, then S is nowhere dense.* ∎

The idea of category makes sense in any topological space since it can be defined in terms of the purely topological notions of the closure and interior of a set. In particular, we can discuss category in the spaces we have made of sets of functions. A set of first category is, in the topological sense, very small. The next theorem tells us that we spent all our time in calculus studying functions selected from a set that, topologically, was hardly there at all! The van der Waerden function is not a rarity at all but typical.[1] Notice, though, that while Theorem 20.9 is quite dramatic, we should be surprised by it only if we assume that the set of continuous functions on an interval is of second category. This is true, but we won't prove it.

THEOREM 20.9: *Among the continuous functions $f : (0, 1) \to \mathbf{R}$, those that are differentiable, even at a single point, form a set of first category.*

PROOF: We will consider only "right-hand" derivatives (the proof gives us a little more than the theorem states). Let S_n be the set of functions f such that, for some $x \in (0, 1 - 1/n)$ and all $h \in (0, 1/n)$, we have $\left|[f(x+h) - f(x)]/h\right| \leq n$. (The choices of x and h serve to guarantee

[1] The word "typical" is used nowadays to mean "except for a set of first category"—hence statements like "The typical real number is irrational." A set whose complement is of first category is called "residual," a curious term, since we normally think of a "residue" as the *smaller* portion of something.

that f is defined at both x and $x+h$.) Any function having a right-hand derivative at any point is certainly in one of these sets. We will show that the sets S_n are closed and that their complements are dense. We will use the sequential characterization of closed sets. Suppose (f_k) is uniformly convergent (that is, convergent in the space of functions). Then $\lim f_k$ is continuous and, since the operation that takes f to $|[f(x+h)-f(x)]/h|$ is continuous, $\lim f_k$ also satisfies the inequality defining S_n (that is, if $f_k \in S_n$ for all k, then $\lim f_k \in S_n$). Thus S_n is closed. According to Lemma 20.8, we may show that S_n is nowhere dense by showing that $\mathbf{R}\backslash S_n$ is dense (that is, that there is an element of $\mathbf{R}\backslash S_n$ close, in the uniform sense, to any continuous function). This is automatically true for any function in $\mathbf{R}\backslash S_n$, and so we need only consider a function $g \in S_n$. Let $\varepsilon > 0$ be given and let r be a function such that $|r(x)| < \varepsilon$ for all x and that any interval contains points x where $|r'(x)| > 2n$ [the function W used in the construction of van der Waerden's function can be adjusted to do this]. Then $g+r \in \mathbf{R}\backslash S_n$ (there are points where $|(g+r)'(x)| > n$) and $g+r$ is uniformly close to g (since $\|(g+r)-g\|_\infty = \|r\|_\infty < \varepsilon$). Thus $\mathbf{R}\backslash S_n$ is dense in the set of continuous functions for each n. ∎

EXERCISES 20.5

1. (a) Prove Lemma 20.8.

   (b) Show that the conclusion of Lemma 20.8 does not hold if the set is not assumed to be closed.

2. (a) Explain why Theorem 20.9 is "only surprising if ... the set of continuous functions on an interval is of second category."

   (b) Does Theorem 20.9 establish that continuous, nowhere differentiable functions exist?

3. Verify the statement "Any function having a right-hand derivative at any point is in one of these sets" made in the proof of Theorem 20.9.

4. Why does the proof of Theorem 20.9 establish *more* than the theorem states?

Chapter 21

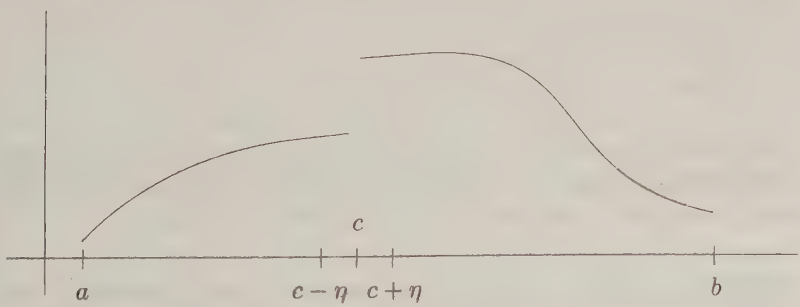
Continuous Functions and Integrability

21.1 INTEGRABLE FUNCTIONS (REVISITED)

We have seen that a continuous function is Riemann integrable, while a function as wildly discontinuous as the Dirichlet function is not. In this chapter we will examine the intermediate ground of this situation and ask how badly a function can fail to be continuous and still be integrable. Though the main result (Theorem 21.6) is extremely important, its complete proof is beyond the scope of this book. Our goal at this time is mainly to observe how a simple result with a simple proof can blossom into something much deeper.

THEOREM 21.1: *If the bounded function $f : [a, b] \to \mathbf{R}$ is continuous except at one point, then f is integrable.*

PROOF: Suppose f is discontinuous at $c \in (a, b)$ (the proof must be adjusted if c is a or b) and that $|f(x)| < M$ for all x. Let $\varepsilon > 0$ be given. Choose a number $\eta > 0$ so that $[c - \eta, c + \eta] \subsetneq (a, b)$ and $M\eta < \varepsilon/12$ (the reason for this choice will be evident shortly).



Since f is continuous on $[a, c - \eta]$ and $[c + \eta, b]$, it is integrable on both

of these intervals. Thus there are partitions P_1 and P_2 of $[a, c-\eta]$ and $[c+\eta, b]$, respectively, so that

$$U(f,P_1) - L(f,P_1) < \varepsilon/3 \quad \text{and} \quad U(f,P_2) - L(f,P_2) < \varepsilon/3.$$

Over $[c-\eta, c+\eta]$ (whose length is 2η), we have $\sup f - \inf f < 2M$. Note that (after an appropriate renumbering), $P = P_1 \cup P_2$ is a partition of $[a,b]$. Then we have

$$\begin{aligned} & U(f,P) - L(f,P) \\ < \; & U(f,P_1) - L(f,P_1) + U(f,P_2) - L(f,P_2) + (2M)(2\eta) \\ < \; & \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ = \; & \varepsilon, \end{aligned}$$

and so f is integrable by Theorem 17.8. ∎

The proof of Theorem 21.1 can be modified to yield:

COROLLARY 21.2: *If $f : [a,b] \to \mathbf{R}$ is continuous except at finitely many points, then f is integrable.*

PROOF: Left as Exercise 21.1.2. ∎

EXERCISES 21.1

1. (a) Draw a picture to illustrate the construction of the partition P in the proof of Theorem 21.1.

   (b) The sketch that accompanies Theorem 21.1 shows a function with a jump discontinuity. Is this used anywhere in the proof? Is the proof valid if the discontinuity is of the second type?

2. Prove Corollary 21.2.

21.2 SETS OF CONTENT ZERO

The proof of Corollary 21.2 hinges on the fact that we can enclose the points of discontinuity of f in a finite collection of intervals whose total length is as small as we wish. We give this property a name:

DEFINITION 21.3: The set S is said to have **content zero** if for any $\varepsilon > 0$ there is finite collection of open intervals $\{I_n\}$, the sum of whose lengths is less than ε and such that $S \subseteq \bigcup_n I_n$.

EXAMPLES 21.2: 1. Any finite set has content 0. If $\varepsilon > 0$ is given and the set has n elements, we can enclose each element in an interval of length $\varepsilon/(n+1)$.

2. $H = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ has content 0. If $\varepsilon > 0$ is given, let $I_1 = (-\varepsilon/4, \varepsilon/4)$. Note that I_1 contains all but finitely many of the elements of H and has length $\varepsilon/2$. We can cover the rest of the elements of H with intervals of total length less than $\varepsilon/2$ as in Example 1.

3. You will show in Exercise 21.2.1 that the set of rational numbers in the interval $[0,1]$ does *not* have content 0 even though, like the set H in the previous example, it is countable. Whether a set has content 0 depends on the way it is distributed on the number line as well as on its cardinality. This is made even more apparent in the next example.

4. The Cantor set (which is uncountable) has content 0. We observed that the sum of the lengths of the open intervals removed in the construction of the Cantor set is 1. If $\varepsilon > 0$ is given, we can stop the process when the sum of the lengths of the (finitely many) intervals removed to that point is greater than $1 - \varepsilon/2$. The Cantor set is contained in a finite collection of *closed* intervals, the sum of whose lengths is less than $\varepsilon/2$. These intervals may be enclosed in turn in open intervals, the sum of whose lengths is less than ε.

The proof of the next theorem is almost the same as that of the last, and so we will gloss over the details.

THEOREM 21.4: *Suppose the bounded function $f : [a,b] \to \mathbf{R}$ is continuous except on a set of content zero. Then f is Riemann integrable.*

PROOF: Suppose $|f(x)| < M$ for all x and let $\varepsilon > 0$ be given. Enclose the points of discontinuity of f in open intervals of total length less than $\varepsilon/6M$. The complement of the union of this collection of intervals is a union of finitely many closed intervals on each of which f is integrable. Find partitions of each of these intervals whose corresponding upper and lower sums are close together. Assemble these partitions and the original collection of intervals into a partition of $[a,b]$ whose corresponding upper and lower sums are close together. ∎

EXERCISES 21.2

1. Show that $\mathbf{Q} \cap [0,1]$ does not have content 0.

2. Supply the details for the proof of Theorem 21.4.

3. We may say that a subset of the plane has "two-dimensional content 0" if it can be contained in finitely many open rectangles, with sides parallel to the coordinate axes, whose total area is as small as we like.

   (a) Draw a picture illustrating this.

   (b) Show that the graph of a uniformly continuous function whose domain is a compact set has content 0.

   (c) Does the result in (b) remain true if the assumptions of uniform continuity or compactness are dropped? (What if the function is continuous but the domain is not compact? Or if the function is continuous but not uniformly continuous?)

   (d) Show that the graph of a Riemann integrable function has content 0.

   (e) Is (d) a consequence of (b)? Vice versa?

   (f) Show that the graph of the Dirichlet function on $[0, 1]$ (see Chapter 19) has two dimensional content 0 according to this definition even though the Dirichlet function is not Riemann integrable.

   (g) The top half of the graph of the Dirichlet function seems to be a copy of the rational numbers, but we saw that $\mathbf{Q} \cap [0, 1]$ does not have content 0. Explain.

   (h) Reconsider these questions with the definition of "two-dimensional content 0" taken to be "can be contained in finitely many open rectangles, with sides parallel to the coordinate axes, the sum of whose *diameters* is as small as we like." (The diameter of a rectangle is the length of a diagonal.)

   (i) Reconsider these questions yet again, with the definition of "two-dimensional content 0" taken to be "can be contained in finitely many sets (of any sort), the sum of whose diameters is as small as we like." [The **diameter** of a nonempty set is the supremum of the collection of distances between points in the set: $D(S) = \sup\{d(x, y) : x, y \in S\}$, where $d(x, y)$ is the distance measurement appropriate to the setting.]

4. Show that the diameter function D defined in the previous problem has the following properties:

   (a) $D(S) \geq 0$ for all S.

   (b) $D(kS) = |k|D(S)$ (see Exercise 5.2.12 for the definition of kS).

   (c) If $S \subseteq T$, then $D(S) \leq D(T)$.

   (d) Give an example of a nonempty set whose diameter is 0.

   (e) Show that it is consistent with these properties to set $D(\emptyset) = 0$.

(f) Show that it is *not* consistent with these properties to assign any value other than 0 to $D(\emptyset)$.

(g) On the other hand, the definition of D given in Exercise 21.2.3.i suggests that $D(\emptyset) = -\infty$. Explain.

(h) Can any general statements be made about the diameters of unions and intersections?

5. (a) The quantity called "two-dimensional content 0" in part (i) of the previous problem is also called "linear content 0." Discuss why this is an appropriate term.

   (b) Suppose S is a subset of the real line, considered as the horizontal axis in the plane. Show that S has content 0 (as a subset of $\mathbf{R}$) if and only if S has linear content 0.

   (c) Show that the result in (b) is not true if we use the definition of "two-dimensional content 0" from the *first* part of the previous problem. In fact, show that every bounded subset of the real line has two-dimensional content 0 under that definition.

21.3 SETS OF MEASURE ZERO

Theorem 21.4 is not the final word on this topic. In Exercise 21.3.7 you will examine a function that is discontinuous precisely on the set of rational numbers, yet is integrable nonetheless. In Definition 21.3, the collection of intervals used must be finite, and this was necessary to make our examples and proofs work. Unfortunately, it obscures the real issue of integrability.

DEFINITION 21.5: The set S is said to have **measure zero** if for any $\varepsilon > 0$ there is a countable collection of open intervals I_n, the sum of whose lengths is less than ε and such that $S \subseteq \bigcup_n I_n$.

Notice that the sum of the lengths of the intervals in Definition 21.5 is actually a series if the collection of intervals is infinite. The requirement of countability in this definition does not restrict our activities much. The sum of any uncountable collection of positive numbers is infinite (see Exercise 13.2.4), and so if the sum of the lengths of the intervals is less than ε, only countably many of them are nonempty. Furthermore, recall that any union of open intervals is an open set and that an open set can be written as a union of *countably* many open intervals (Theorem 8.11).

EXAMPLES 21.3: 1. Enumerate the rational numbers: $r_1, r_2, \ldots$. Let $\varepsilon > 0$ be given and let $I_n = (r_n - \varepsilon/2^{n+2}, r_n + \varepsilon/2^{n+2})$. Then $\mathbf{Q} \subseteq \bigcup_n I_n$,

the length of I_n is $\varepsilon/2^{n+1}$, and the sum of the lengths of the intervals I_n is $\varepsilon/2 < \varepsilon$. Thus the set of rational numbers has measure zero (it is small in this sense despite the fact that it is dense). This proof can be modified to show that *any* countable set has measure zero.

The following theorem *does* tell the whole story of Riemann integrability, but its proof would take us too far afield. The proof of Theorem 21.4 relied on the fact that the complement of a union of finitely many intervals is again a union of finitely many intervals. This is not so if the former collection is not finite. (The Cantor set is the complement of a union of countably many intervals, but contains no interval.)

THEOREM 21.6: *A function $f : [a, b] \to \mathbf{R}$ is Riemann integrable if and only if the set of points at which f is discontinuous has measure zero.* ∎

EXERCISES 21.3

1. Show that any set of content 0 also has measure 0.

2. Find an irrational number not in the union of the intervals used to show that the rational numbers have measure zero.

3. Show that any countable set has measure zero and is of first category.

4. Show that any subset of a set of measure zero has measure zero.

5. Show that a countable union of sets of measure zero has measure zero.

6. (a) We might define the length of an open set to be the sum of the lengths of the open intervals in the representation of the set given in Theorem 8.11 (this might, of course, be infinite). Show that the length of a union of open sets is not greater than the sum of the lengths of the sets.

   (b) Show that the length of a union of *disjoint* open sets is equal to the sum of the lengths of the sets.

   (c) Show that a set has measure 0 if and only if, given $\varepsilon > 0$, it can be contained in an open set whose length is less than ε.

7. Define a function on $[0, 1]$ by $f(x) = 1/n$ if $x = m/n \in \mathbf{Q}$ is in lowest terms and $f(x) = 0$ if $x \notin \mathbf{Q}$. Show that f is continuous at each irrational number and discontinuous at each rational number. By Theorem 21.6, this means that f is integrable. Find its integral.

8. We have seen several ways in which a set can be considered to be small. Cardinality, category, content, and measure all provide us with ideas of smallness. Discuss the relationships among these concepts. For instance, a countable set must have measure zero and also must be of first category but need not have content zero. On the other hand the fat Cantor set is a set that is of first category whose measure is not zero. An examination of all possible combinations is a major undertaking.

21.4 A SPECULATIVE GLIMPSE AT MEASURE THEORY

The idea of measure 0 is a hint of the much larger topic of measure theory. One of the first big surprises of measure theory is that, while it is very easy to define "measure 0," it is very difficult to define "measure." Even so, we can guess at some of the ideas of measure theory by thinking about a topic with considerably more intuitive familiarity: Probability.

If we select an element of $[0,1]$ at random (though it is not at all clear what this means or whether it is possible in any practical sense), there is, it seems, a probability of $1/2$ that it will be in $[0,1/2]$ because the length of $[0,1/2]$ is half that of $[0,1]$. If a subset of $[0,1]$ is open (so its length may be computed as in Exercise 21.3.6), the probability that a randomly chosen element is in that set should be the length of the set.[1] If two open sets are disjoint, the length of their union is the sum of their lengths. This might lead us to make the following guess. Unfortunately, *we aren't prepared to decide whether this guess is true or not.* That's why it's a guess! Sorting this out is one of the first major issues in the study of measure theory. We will say something in a moment that will cast doubts on the whole enterprise.

GUESS 21.7: *If $A \cap B = \emptyset$, the probability of selecting an element of $A \cup B$ is the sum of the probability of selecting an element from A and the probability of selecting an element from B (in other words, the measure of a disjoint union is the sum of the measures of the two sets).*

If one set is contained in another, the probability of selecting an element of the smaller one should be no larger than that of selecting an element of the larger one. The definition of "measure 0" is that the set in question can be contained in an open set of arbitrarily small length. This leads to the following (correct) guess:

[1] To be more precise, this probability is the length of the subset divided by the length of the whole set from which the number is chosen.

GUESS 21.8: *If a number is selected from $[0,1]$ at random and $S \subseteq [0,1]$ has measure 0, the probability that the element is in S should be 0.*

(If this guess is correct, the probability that a randomly selected number is *rational* is 0.) We have been very conservative in Guess 21.8 since we know the meaning of "measure 0." We might have been lead by the preceding discussion to make the following *incorrect* guess:

GUESS 21.9: *If $A \subseteq B \subseteq [0,1]$, then the probability that a randomly selected element is in A is not larger than the probability that it is in B.*

How could this fail to be true? We now come to the only real result of this section.[2] This theorem, as we have said, throws a shadow over *any* guesses we might make concerning probability or measure and warns us that the subject is not easy. We will assume that Guess 21.7 holds for countable unions under "suitable conditions" (what these conditions might be, we can only guess).

THEOREM 21.10: *Under the assumptions mentioned above, there exists a nonmeasurable set (a set to which it is impossible to attach a probability).*

PROOF: We will gather the elements of the interval $[-1/2, 1/2]$ into a collection of classes in this way: Two numbers will be in the same class if they differ by a rational number (for instance, all the rational numbers in the interval are in one class, $\pi/9$ is in a different class, and $\pi/9 + 1/37$ is in the same class as $\pi/9$). These classes are clearly disjoint, and each of them is countable. There must be uncountably many of these classes since their union is the uncountable set $[-1/2, 1/2]$. Let $\mathcal{B}$ be a set that consists of one element chosen from each of these classes. We will show that $\mathcal{B}$ can't be measurable. Each element of $[-1,1]$ differs from some element of $[-1/2, 1/2]$ by a rational number in $[-1/2, 1/2]$. It follows that the union of translations[3] of $\mathcal{B}$ by rational elements of $[-1/2, 1/2]$ is $[-1,1]$. Each of these translations should have the same measure as $\mathcal{B}$. If the measure of $\mathcal{B}$ is 0, then this (countable) union must also have measure 0. But $[-1,1]$ doesn't have measure 0 (its measure would seem to be 2). Perhaps $\mathcal{B}$ has some positive measure, say β. Then it must be that $2 = \beta + \beta + \beta + \cdots$, which is impossible. This contradicts the

[2] Keep in mind that the object of these selected shorts is to explore how *bad* things can get!

[3] Recall that the translation of a set S by a number x is denoted $S + x$ and is given by: $S + x = \{y : \exists s \in S \ni (y = s + x)\}$.

assumption that the rules of measure theory (which we have only guessed at!) apply to this set $\mathcal{B}$. We must conclude that $\mathcal{B}$ is *nonmeasurable.* ∎

With Theorem 21.10 in mind, we can be a little more cautious and update Guess 21.9 (which was incorrect) to the following (which is correct):

GUESS 21.11: *If $A \subseteq B$ and both A and B are measurable (if a probability can be assigned to them), then the measure of A is not more than the measure of B.*

Thinking about probability some more, if we specify a subset of $[0,1]$ and choose an element at random, it seems that the probability that the chosen number is in the set should be

$$[1 - (\text{the probability that it is not})].$$

This leads us to the following (correct) guess:

GUESS 21.12: *If a set is measurable, its complement is measurable.*

Now we are really getting somewhere. We suspect very strongly that open sets are measurable and that we can find their measures (Exercise 21.3.6). Guess 21.12 would mean that closed sets are also measurable How would we find the measure of a closed set? A closed set is not necessarily a union of intervals like an open set, and so that method won't work. Suppose for the moment that our closed set can be contained in some interval (a, b). Then its measure should be

$$[(b-a) - (\text{the measure of its (open) complement})].$$

This is well and good, but not every closed set (nor every measurable set) is bounded. Notice that we do have all the machinery needed to check the property in the next definition.

DEFINITION 21.13: The set S is **essentially bounded** if there is an open, bounded interval (a, b) so that $S \backslash (a, b)$ has measure 0.

GUESS 21.14: (a) *A set that is not essentially bounded is either nonmeasurable or has infinite measure.*

(b) *If S is an essentially bounded, closed set and (a, b) is as in the definition, the measure of S is $(b-a) - [$the measure of $(a,b)\backslash S]$.*

Now things are beginning to come together. In Exercise 8.3.2 we saw how we can approximate a set from the inside with an open set (its interior) and from the outside with a closed set (its closure). In view of Guess 21.11, such an approximation would give us an estimate of the

measure of a set. Unfortunately, this is not what we want, for a very simple reason: It doesn't work! The closure of $[0,1] \cap \mathbf{Q}$ is $[0,1]$. We know that $[0,1] \cap \mathbf{Q}$ has measure 0, but its closure seems to have measure 1. This is not a very good approximation. The trick is to turn "interior" and "closure" on their heads.

DEFINITION 21.15: (a) The **outer measure** of a set S is the infimum of the measures of all open sets *containing* S.

(b) The **inner measure** of S is the supremum of the measures of all essentially bounded, closed sets *contained in* S.

(c) A set is **measurable** if its inner measure and outer measure are the same, in which case the **measure** of the set is this common value.

Now this has a nice familiar ring to it. Not only does it look a lot like many of the things we've done in the past, but the computation of an outer measure is very much like the process by which we determine that a set has measure 0. You stand poised and ready to leap into the study of measure theory in much the way that you were ready to leap into real analysis when you began this text. You begin this journey in the company of an old friend, a variant of the Cauchy criterion.

THEOREM 21.16: *A bounded set S is measurable if and only if given $\varepsilon > 0$ there is a closed set A and an open set B such that $A \subseteq S \subseteq B$ and the difference between the measures of A and B is less than ε.* ∎

EXERCISES 21.4

1. (a) Suppose that A and B are measurable sets with the property that there are disjoint open sets C and D with $A \subseteq C$ and $B \subseteq D$. (This is stronger than simply saying A and B are disjoint.) Show that the measure of $A \cup B$ is the sum of the measures of A and B.

   (b) Give examples of disjoint sets that *don't* have the property in (a).

   (c) Discuss Guess 21.7.

2. (a) Several assumptions about measure and probability are made in the proof of Theorem 21.10. Find them and discuss whether they are reasonable.

   (b) Verify that every element of $[-1,1]$ differs from some element of $[-1/2,1/2]$ by a rational element of $[-1/2,1/2]$.

   (c) A *really* big assumption about set theory was made in the proof of Theorem 21.10. It is found in the statement "Let $\mathcal{B}$ be a set that

consists of one element from each of these sets." Discuss what is being assumed here and whether it is reasonable.[4]

3. Consider the set given by $S = \bigcup_{n=1}^{\infty} \left[n, n + \frac{1}{2^n}\right]$

   (a) Show that S is closed.

   (b) Show that S is not essentially bounded.

   (c) Explain why it is reasonable to say that the measure of S is 1.

   (d) *Prove* that the measure of S is 1, using Definition 21.15.

   (e) Discuss Guess 21.14.a.

4. Prove Theorem 21.16.

5. (a) Suppose S has content 0 and that $f : \mathbf{R} \to \mathbf{R}$ is a differentiable function whose derivative is bounded. Show that $f(S)$ has content 0.

   (b) Show that this remains true if f is only assumed to be absolutely continuous (see Exercise 17.7.7).

   (c) Show that (a) is true with "content" replaced by "measure."

   (d) Let K be the Cantor function (constructed in Chapter 19) and C the Cantor set (which has measure 0). Show that $K(C) = [0, 1]$, and consequently that *the continuous image of a set of measure 0 can have positive measure* (small can be continuously deformed into big!). What does this say about the derivative of the Cantor function? What does this say about the possibility that the Cantor function is absolutely continuous?

6. (a) Recall the brief look at Lebesgue integration we took in Exercise 17.7.8. Show that the Dirichlet function is Lebesgue integrable, and that its integral is 0.

   (b) Assuming for the moment that any Rieman integrable function is also Lebesgue integrable, discuss whether a function that *differs* from a Riemann integrable function on a set of measure 0 is necessarily Lebesgue integrable.

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[4] WARNING!! There is *much* more involved here than meets the eye. If you convince yourself that it is always possible to select an element from each of a collection of nonempty sets (without the use of some formula), read "The Banach-Tarski paradox" [Karl Stromberg, *American Mathematical Monthly* 86 (1979), 151–161]. The third paragraph alone will be enough to shake your confidence.

Chapter 22

We Build the Real Numbers

22.1 DO THE REAL NUMBERS REALLY EXIST?

We have spent a good bit of time thinking about the field of real numbers and have seen six manifestations of its property of completeness, which distinguishes it from other ordered fields. What we have *not* done is show that it exists! None of what we've done has anything to say about the question:

Is there an Archimedean ordered field that
has the Least Upper Bound property?

In this chapter, we will show that such a things exists in the most convincing way possible: We will build one. Here is what we must do:

(1) Define the phrase "real number."
(2) Define addition and multiplication.
(3) Show that this structure is a field.
(4) Define a positive set.
(5) Show that the field is complete.

22.2 DEDEKIND CUTS

DEFINITION 22.1: A subset C of the rational numbers is called a **Dedekind cut** (or simply a **cut**) if:

(i) $C \neq \mathbf{Q}$ and $C \neq \emptyset$.
(ii) if $p \in C$ and $q < p$, then $q \in C$.
and (iii) if $p \in C$, there is a $q > p$ with $q \in C$.

EXAMPLES 22.2: 1. It is easy to check that $\{p \in \mathbf{Q} : p < 1/4\}$ is a cut and that $\{p \in \mathbf{Q} : p \leq 1/4\}$ is not [condition (iii) is violated].

2. $A = \{p \in \mathbf{Q} : p > 0 \text{ and } p^2 < 2\} \cup \{p \in \mathbf{Q} : p \leq 0\}$ is a cut.
(i) $0 \in A$, and so $A \neq \emptyset$; $2 \notin A$, and so $A \neq \mathbf{Q}$.

(ii) Suppose $p \in A$ and $q < p$. If $q \leq 0$, then $q \in A$. If $q > 0$, then $0 < q < p$, and so $0 < q^2 < p^2 < 2$ and $q \in A$.

(iii) Let $p \in A$. If $p \leq 0$, then $p < q = 1 \in A$. Suppose $p > 0$. As in the proof of Theorem 5.4, we may find a positive rational number, r, with $(p+r)^2 < 2$. Then $q = p + r$ fulfills the definition. (If p is rational, so is $r = (2 - p^2)/7$.)

For any rational number p, let $p^\star = \{q \in \mathbf{Q} : q < p\}$. It is clear that $p^\star$ is a cut. These are called the **rational cuts**. The first set in Example 1 is a rational cut, but even though the set A in Example 2 is a cut, it is not a rational cut. We will see shortly that A is actually $\sqrt{2}$.

EXERCISE[1] I: Show that if C is a cut, $p \in C$, and $r \notin C$, then $p < r$. (Remember this as *Anything outside a cut is larger than anything inside.*)

EXERCISES 22.2

1. Show that the union of any set of cuts is either a cut or is all of $\mathbf{Q}$.

2. (a) Check that the reference to the proof of Theorem 5.4 is valid.

   (b) Show that if p is a rational number, then so is $r = (2 - p^2)/7$.

22.3 THE ALGEBRA OF CUTS

DEFINITION 22.2: If A and B are cuts, we define their sum $\boldsymbol{A + B}$ by: $A + B = \{r : r = p + q \text{ for some } p \in A \text{ and } q \in B\}$.

THEOREM 22.3: *If A and B are cuts, then $A + B$ is a cut.*

PROOF: (i) A and B are nonempty, and so $A + B$ is nonempty. Let p_0 and q_0 be in the complements of A and B, respectively. By Exercise I, $p_0 > p$ and $q_0 > q$ for all $p \in A$ and $q \in B$. Then $p_0 + q_0 > p + q$ for all such p and q, and so $p_0 + q_0 \notin A + B$. Thus $A + B \neq \mathbf{Q}$.

(ii) Let $r \in A + B$, $s < r$, and $d = r - s \in \mathbf{Q}$. Since $r \in A + B$, there are elements p and q of A and B with $r = p + q$. Now $s = r - d = (p - d) + q \in A + B$ since $p - d \in A$.

(iii) Let $r = p + q \in A + B$, and $p < p_0 \in A$. Then $r < p_0 + q \in A + B$. ∎

---

[1] Exercises I, II, and III, which are in the text of this chapter, play such an important role in the development that they should be done as they appear.

THEOREM 22.4: *Addition of cuts obeys rules* (1) *through* (4) *of the definition of a field.*[2]

PROOF: Addition is clearly commutative and associative. We might guess that 0^* acts as a zero element (this cut was defined just before Exercise I). We must show that $A + 0^* = A$ for any cut A. Here we begin to see the usefulness of having constructed the real numbers as we have. We must establish equality of two *sets*, and so we have all of those techniques available to us. If $p \in A$ and $z \in 0^*$, then $z < 0$, and so $p + z < p$ and $p + z \in A$. Thus $A + 0^* \subseteq A$. To establish the other inclusion, we must show that any element of A can be written as the sum of an element of A and a negative number (an element of 0^*). Let $p \in A$ and $r > 0$ with $q = p + r \in A$ (be sure you see why such an r exists). Then $p = q + (-r) \in A + 0^*$, and so $A \subseteq A + 0^*$. (*The proof is not yet complete, but we return now to the discussion.*)

The question of additive inverses is more delicate. We must define $-A$ so that $A + (-A) = 0^*$. In other words, the sum of an element of A and an element of $-A$ must be negative. We might make a guess:

$$???? \qquad -A = \{p \in \mathbf{Q} : p + q < 0 \text{ for all } q \in A\} \qquad ????$$

But this *doesn't work*. If this were our "definition," we would have $-(1^*) = \{p \in \mathbf{Q} : p \leq -1\}$, which is *not a cut*.

It seems reasonable to expect that $-(1^*) = (-1)^*$ (the set above without the element -1). Perhaps we can adjust our definition to exclude -1. Does -1 behave differently in this construction than other elements? When we "add" -1 to 1^*, we get: $-1 + \{p : p < 1\} = \{p : p < 0\}$. On the other hand, if $q < -1$, we have $q + \{p : p < 1\} = \{p : p < 1 + q\}$. Note that $1 + q < 0$. For all elements p in $\{p \in \mathbf{Q} : p \leq -1\}$ *except* -1, the supremum of $p + 1^*$ is negative, where for -1 the supremum is 0. Evidently $-(1^*)$ should consist of all q such that $\sup(q + 1^*) < 0$. We have to work around the fact that we can't really say "supremum" yet.

DEFINITION 22.5: For a cut A, let $-A$ be defined by

$$-A = \{p \in \mathbf{Q} : \exists r_p < 0 \ni \forall q \in A (p + q < r_p)\}.$$

THEOREM 22.6: *If A is a cut, $-A$ is a cut.*

PROOF: (i) Suppose $p^\# \notin A$ and let $p = -p^\# - 1$. For any $q \in A$ we have $q + p = q - p^\# - 1 < -1$ since $p^\# > q$. So $p \in -A$ (we may take

[2] The proof of Theorem 22.4 extends all the way to the end of the proof of Theorem 22.7. We will need some intermediate results to complete it.

$r_p = -1$), and so $-A \neq \emptyset$. If $q \in A$, then $-q \notin -A$ (since $q + -q = 0$), and so $-A \neq \mathbf{Q}$.

(ii) Let $p \in -A$ and $r_p < 0$ be as in the definition. If $p_0 < p$, then $p_0 + q < p + q < r_p$ for all $q \in A$, so that $p_0 \in -A$ (that is, we can take $r_{p_0} = r_p$).

(iii) Let p and r_p be as above and let $p_1 = p - (1/2)r_p$. Note that $p_1 > p$ since $r_p < 0$. For any $q \in A$, we have

$$q + p_1 = q + p - (1/2)r_p < r_p - (1/2)r_p = (1/2)r_p,$$

and so we can take $r_{p_1} = (1/2)r_p$ to show that $p_1 \in -A$. ∎

THEOREM 22.7: *For any cut A, we have $A + (-A) = 0^\star$.*

PROOF: Let $q \in A$ and $p \in -A$. Then $-p > q$ since $-p \notin A$. So $q + p < 0$; that is, $q + p \in 0^\star$, and $A + -A \subseteq 0^\star$. The other inclusion is trickier. Let $p \in 0^\star$. We must write p as the sum of an element of A and an element of $-A$. Let $r = -(1/2)p$. Now $r > 0$, and so there is an integer n so that $nr \in A$ but $(n+1)r \notin A$ (since the rational numbers have the Archimedean property). Let $s = -(n+2)r$. Then $s \in -A$ since for any $q \in A$ we have:

$$\begin{array}{rl} & q + s \\ = & q - (n+2)r \\ = & q - (n+1)r - r \\ < & -r \\ < & 0 \end{array}$$

[since $(n+1)r \notin A$, it is larger than q]. Finally, $p = nr + s \in A + -A$, and so $0^\star \subseteq A + -A$. ∎∎ (This also completes the proof of Theorem 22.4.)

We have found an additive structure for cuts. We now turn our attention to multiplication. When we were young, we were taught first how to multiply positive numbers, and then we were shown how to find the sign of a product if it involved negative numbers. (We were never really taught multiplication for negative numbers!) We will use the same approach here, but we must be a bit careful because "positive" doesn't have any meaning to us yet.

DEFINITION 22.8: Let A and B be cuts such that $0 \in A \cap B$. We define their product ***AB*** by

$$AB = \{pq : p \in A, q \in B, p > 0, \text{ and } q > 0\} \cup \{p : p \leq 0\}.$$

[By part (iii) of the definition of a cut, any cut that contains 0 also

contains positive rational numbers. These cuts will ultimately become the positive real numbers.]

THEOREM 22.9: *If A and B are cuts with $0 \in A \cap B$, then AB is a cut.*

PROOF: (i) Follows as in the proof for addition.

(ii) Since $\{p : p \leq 0\} \subseteq AB$ by definition, we need only check the case where $r = pq \in AB$ with $p, q > 0$, and $0 < s < r$. Write $s = (s/r)r = [(s/r)p]q$. Now $s/r < 1$, and so $p > (s/r)p \in A$. Since $q \in B$, we have written s as the product of a positive element of A and a positive element of B, and so $s \in AB$.

(iii) Let $r \in AB$. If $r \leq 0$, then for any $0 < p \in A$ and $0 < q \in B$, we have $r \leq 0 < pq \in AB$. If $r = pq$, where $0 < p \in A$ and $0 < q \in B$, and $p < p_0 \in A$, we have $r < p_0 q \in AB$. ■

Multiplication (when it is defined) is clearly associative and commutative. To extend the definition of multiplication to all cuts, we first prove the following, which will become the trichotomy when we define the positive set:

THEOREM 22.10: *For any cut A, exactly one of the following holds:*

$$\begin{array}{rl} & \text{(i)}\ \ 0 \in A; \\ & \text{(ii)}\ \ A = 0^\star; \\ \text{or} & \text{(iii)}\ \ 0 \in -A. \end{array}$$

PROOF: We will show first that no two of these can happen at the same time. This is clear for the pair (i) and (ii) and the pair (ii) and (iii) (since $0^\star$ is its own additive inverse). Suppose that (i) and (iii) both occur for some cut A. Then $0 = 0 + 0 \in A + -A = 0^\star$, which is a contradiction since $0 \notin 0^\star$. Now we show that one of these must occur. Suppose that both (i) and (ii) fail for the cut A. Since $0 \notin A$, there is some $p < 0$ with $p \notin A$. For any $q \in A$ we then have $q + 0 = q < p < 0$, so that $0 \in -A$. ■

DEFINITION 22.11: For cuts A and B, let

$$AB = \begin{cases} AB & \text{(as above) if } 0 \in A \cap B \\ -(A(-B)) & \text{if } 0 \in A \backslash B \text{ and } B \neq 0^\star \\ (-A)(-B) & \text{if } 0 \notin A \cup B \text{ and } A, B \neq 0^\star \\ 0^\star & \text{if } A = 0^\star \text{ or } B = 0^\star. \end{cases}$$

Notice that the products that are computed in this definition involve

only cuts that contain zero. Since we have already established that the product of two such cuts is a cut, and that the additive inverse of a cut is a cut, we need not establish these results again. The definition of the multiplicative inverse and the associated proofs are much like those for the additive inverse and are omitted.

THEOREM 22.12: *The set of cuts is a field.* ∎

DEFINITION 22.13: The **real numbers**: $\mathbf{R} = \{A \subseteq \mathbf{Q} : A \text{ is a cut}\}$.

EXERCISE II: Let A and B be cuts. Show that either $A \subseteq B$ or $B \subseteq A$.

This is more significant than it appears. It will play an important role in the definition of the ordering on the real numbers.

EXERCISES 22.3

1. Show that $-(1^\star) = \{p \leq -1\}$ under the "guessed" definition.
2. Verify that addition and multiplication are associative and commutative.
3. (a) Show that $1^\star + 2^\star = 3^\star$ and that $p^\star + q^\star = (p+q)^\star$ for any p and q.

   (b) Show that $p^\star q^\star = (pq)^\star$ for any rational numbers p and q. (If you have taken algebra, you recognize that this exercise says the rational numbers are *isomorphic* to the rational cuts.)
4. Show that the cut A in Example 21.1.2 is $\sqrt{2}$; that is, show that $AA = 2^\star$. (Caution! The elements of AA are *not* just the squares of elements of A.)

22.4 THE ORDERING OF CUTS

DEFINITION 22.14: The **positive set** of real numbers is given by

$$P = \{A \in \mathbf{R} : 0 \in A\}.$$

We have already established the trichotomy. The other properties of the positive set are easy to show.

EXERCISE III: Show that $A \leq B$ if and only if $A \subseteq B$.

This highlights the dual nature of the real numbers and establishes a geometric fact you probably suspected already. The real numbers are elements of an ordered field, but they are also sets.

EXERCISES 22.4

1. Verify that P is a positive set.

22.5 THE CUTS ARE THE REAL NUMBERS!

All the preceding work leads to the following, which says that the set of real numbers, as defined in this chapter, has one of the properties of the Big Theorem (and hence have them all). Note well: This is now a *theorem*, not an *axiom*.

THEOREM 22.15: $\mathbf{R}$ *has the Least Upper Bound property.*

PROOF: Let $S = \{A_\alpha : \alpha \in \mathcal{A}\}$ be a nonempty set of real numbers that is bounded above. (Other than the requirement that $\mathcal{A} \neq \emptyset$, the nature of the index set $\mathcal{A}$ is unimportant.) Since S is bounded above, there is a real number A so that $A_\alpha \leq A$ for all $\alpha \in \mathcal{A}$. By Exercise III, this means $A_\alpha \subseteq A$ for all $\alpha \in \mathcal{A}$. Let $B = \bigcup_{\alpha \in \mathcal{A}} A_\alpha$. It is easy to see that B is a real number and that it is an upper bound for S. Suppose C is an upper bound for S; that is, $A_\alpha \leq C$ ($A_\alpha \subseteq C$) for all α. Then $B = \bigcup_{\alpha \in \mathcal{A}} A_\alpha \subseteq C$; in other words, $B \leq C$. So B is the least upper bound of S. ∎

The brevity of this proof is a good example of an important pattern in mathematics. The work in this chapter has gone into picking the right *definitions*. Having done this well, the important theorems can be proved easily.

EXERCISES 22.5

1. Since $\mathbf{R}$ is linearly ordered, we may define cuts of real numbers (the definition is the same as for cuts in $\mathbf{Q}$).

   (a) Show that if C is a cut of real numbers, then C is bounded above and $C = \{x \in \mathbf{R} : x < \sup C\}$. An ordered field $\mathbf{F}$ in which every cut is of the form $\{x \in \mathbf{F} : x < a\}$ for some $a \in \mathbf{F}$ is said to have the **Dedekind property** (note that $\mathbf{Q}$ does *not* have the Dedekind property).

(b) In (a) you showed that the Least Upper Bound property implies the Dedekind property. (The Least Upper Bound property guarantees that C has a supremum.) Now show that any ordered field with the Dedekind property also has the Least Upper Bound property. (That is, the Dedekind property is part of the Big Theorem.)

2. Suppose $S \subseteq \mathbf{R}$ is bounded above. Let
$$U = \{y : y < x \text{ for some } x \in S\}.$$
Show that U is a cut and that $\sup S = \sup U$.

3. The phrase "limit superior" has been defined three times in this book for three different sorts of objects: Collections of sets (Exercise 1.15.9), individual sets (Exercise 7.6.6), and sequences of numbers (Exercise 10.4.10). Use the definition of a real number as a Dedekind cut to relate these definitions to each other. For example, if each term in a sequence (x_n) is considered to be a cut, is the limit superior of (x_n) (a collection of sets) the same as the limit superior of (x_n) (a sequence of numbers)?

4. We could also construct the real numbers from Cauchy sequences of rational numbers. The first problem we encounter is that a number can be the limit of many sequences. If $X = (x_n)$ and $Y = (y_n)$, recall that we have defined the weave of X and Y to be $X \varpi Y = x_1, y_1, x_2, y_2, \ldots$ (see Exercise 10.4.13). Let us say the sequence X is equivalent to the sequence Y if $X \varpi Y$ is a Cauchy sequence. Show that this is an equivalence relation (see Exercise 2.1.5). Now let $\mathbf{R}$ be the collection of equivalence classes of such sequences and follow the outline of this chapter to show that $\mathbf{R}$, so defined, is an Archimedean ordered field in which the Cauchy criterion holds. In view of the Big Theorem, all the other properties of the real numbers discussed in Part 2 of the book also hold for this field. Note that if one takes this approach, the Archimedean property must be established separately, since it is not implied by the Cauchy criterion.

5. Here we investigate whether the field in Exercise 22.5.4 is really the same as the one constructed in this chapter.

(a) Show that any field with the Least Upper Bound property must contain a copy of the real numbers. (We've seen before that any ordered field contains a copy of the rational numbers.)

(b) Find a part of the Big Theorem that must be violated if such a field has any elements other than those found in (a).

(c) Does this answer the question at the beginning of the exercise?

6. In this chapter, we constructed the real numbers from a much more familiar set, the rational numbers. In this exercise we will construct the integers from the (more familiar) natural numbers. In the next exercise, we will construct the rational numbers from the (more familiar) integers. We will assume that we know how to (i) add two natural numbers, (ii) recognize when one natural number is larger than another, and (iii) subtract a smaller natural number from a larger one (but that, in general, we do not know how to subtract natural numbers). Consider the set consisting of the symbols 0, $(+, n)$, and $(-, n)$ for $n = 1, 2, \ldots$. (In the end, we will think of the symbols $(+, n)$ as the positive integers, and the symbols $(-, n)$ as the negative integers.)

   (a) Define addition and multiplication of these objects. Remember that your definition can involve operations defined only on natural numbers.

   (b) Show that the addition and multiplication you have defined are commutative.

   (c) Show that every element of this set has an additive inverse.

   (d) Show that multiplication distributes over addition.

   (e) Show that not every element of this set has a *multiplicative* inverse.

   (f) Define an ordering on this set. (The set of positive elements—those that are greater than 0 in your ordering—should have the same properties as the positive set in an ordered field: It should be nonempty; the sum of two positive elements should be positive; the product of two positive elements should be positive; and the trichotomy should hold.)

7. Here we construct the rational numbers from the integers.

   (a) Consider the set of ordered pairs (p, q), where p and q are integers and $q \neq 0$. While we might want to consider (p, q) as being the same as the "quotient" p/q, we can't do so. Why? Consider the (different) pairs $(1, 2)$ and $(3, 6)$.

   (b) Define a relation $\approx$ between these pairs by $(p, q) \approx (m, n)$ if $pn = qm$. Show that this is an equivalence relation (see Exercise 2.1.5).

   (c) Denote the equivalence class of the pair (p, q) by $[p, q]$. We will define addition of these equivalence classes by
$$[p, q] + [m, n] = [pn + qm, qn].$$
   Show that this operation yields a valid equivalence class.

   (d) Show that the operation defined in (c) is **well-defined**, that is, if $(p, q) \approx (r, s)$ and $(m, n) \approx (j, k)$, then $[p, q] + [m, n] = [r, s] + [j, k]$.

["Well-defined" is a deceptive phrase. It appears to mean something very general when, in the mathematical sense, it means something very specific: Whenever you define an operation on equivalence classes by referring to *representatives* of those classes (as we have done here), you must ask whether the result would be the same if you had begun with a different representative of each class.]

(e) This collection of equivalence classes of ordered pairs of integers is what we will call the **rational numbers**. Define multiplication on this set and show that the resulting structure is a field.

(f) Define a positive set on the rational numbers and show that the result is an ordered field.

8. Define "ordered pair" without using any of the words "first," "second," "left," "right," or any other "directional" indicator. Whatever your definition is, it has to be the case that $(a, b) = (c, d)$ if and only if $a = b$ and $c = d$.

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MATHEMATICS

INTRODUCTION TO REAL ANALYSIS

Michael J. Schramm

This text forms a bridge between courses in calculus and real analysis. It focuses on the construction of mathematical proofs as well as their final content. Suitable for upper-level undergraduates and graduate students of real analysis, it also provides a vital reference book for advanced courses in mathematics.

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